Remarks on reducts of varieties

Research supported by NRC Grant A7256 and a Summer Research Institute Grant.

Stanley Burris

There are a number of interesting classes of algebras which, although not themselves varieties, can be advantageously studied as reducts of varieties. In § 1 we look at strict Horn classes and reducts of varieties, with special attention to the possibility of finding constructive methods of answering some fundamental questions. This leads to § 2 where we examine when a quasi-variety is actually a variety. Finally in § 3 certain strict Horn classes are shown to be homomorphism-preserving reducts of varieties. The examples presented in this paper are not new, but the approach via Skolemization, etc. offers a unifying perspective.

§ 1. Strict Horn Classes

A class $K$ of algebras is a reduct of a class $K'$ of algebras (or $K'$ is an expansion of $K$) if $K$ is obtained from $K'$ by forgetting some of the fundamental operations. For example Abelian groups form a reduct of the class of rings. Unfortunately a reduct of a variety may not even be an elementary class (Kogalovskii [10]). Indeed, the situation is even worse, as we shall see in Theorem 2. The negative results on algorithms (such as this one) are based on Lemma 1. If $\Pi$ is a finite set of semigroup relations with an unsolvable word problem and $\cdot$ is the semigroup operation let $\Pi_{ab}$ be the set of equations $\Pi \cup \{x \cdot (y \cdot z) = (x \cdot y) \cdot z, x \oplus a = x, x \oplus b = y \oplus b\}$ (in the language with two binary operations $\cdot$ and $\oplus$, and with nullary operations consisting of the generators of the semigroup belonging to $\Pi$), where $a$ and $b$ are words in the original semigroup.

Lemma 1. There is no algorithm to determine if $\Pi_{ab} \vdash x = y$.

1) An excellent survey of such decidability questions is given in McNulty [13].
**Proof.** It is not difficult to see that $\Pi_{ab} \vdash x = y \iff \Pi \vdash a = b$, and the latter is an undecidable question. \hfill \Box

**Theorem 2.** There is no algorithm by which one could determine from any finite set of equations $\Sigma$ and any reduct of the variety defined by $\Sigma$ whether or not this reduct is an elementary class.

**Proof.** Let $\Sigma_R$ be a finite set of equations defining rings, and form $\Sigma = \Pi_{ab} \cup \Sigma_R$ (we assume that the fundamental operations symbols of $\Pi_{ab}$ and $\Sigma_R$ are distinct). Let $K$ be the reduct of the variety defined by $\Sigma$ obtained by dropping all the ring operations except multiplication. If $\Sigma \vdash x = y$ then any infinite multiplication semigroup which is a reduct of a ring will also be a reduct of an algebra in $K$ as every ring $R$ can be arbitrarily juxtaposed on any model of $\Pi_{ab}$ of the same cardinality to obtain a model of $\Sigma$ (because the operation symbols in $\Pi_{ab}$ and $\Sigma_R$ are distinct). However, the class of multiplicative semigroups which are reducts of rings do not form an elementary class (see [10]), so there is an infinite semigroup $S$ which is not a reduct of any ring, but it is elementarily equivalent to a reduct of a ring. By Shelah [17] there is an index set $I$ and ultrafilter $\mathcal{U}$ such that $S^I/\mathcal{U}$ is a reduct of a ring (note that the multiplicative semigroup reducts of rings are closed under ultrapowers). Choose a model $A$ of $\Pi_{ab}$ of the same cardinality as $S$, and juxtapose $S$ arbitrarily on $A$ to form $B$. Then $B^I/\mathcal{U} \in K$. Now $B \models \Sigma$, hence $K$ cannot be an elementary class. But then clearly $K$ will be an elementary class iff $\Sigma \vdash x = y$, and the latter obviously holds iff $\Pi_{ab} \vdash x = y$, and by Lemma 1 there is no algorithm for this last question. \hfill \Box

If, nonetheless, we restrict our attention to those reducts of varieties which do form elementary classes we have a rather satisfying result. First recall (see [5]) that a strict Horn sentence (in prenex form) looks like

$$Q_1 x_1 \ldots Q_n x_n \left( \forall \left( \bigwedge_{j=1}^l (Q_j p_{ij} = q_{ij}) \rightarrow p_j = q_j \right) \right),$$

where each $Q_i$ is either $\forall$ or $\exists$. If there is exactly one $j$-index and all the quantifiers are universal it is called a quasi-identity (see [11]).

**Theorem 3.** For an elementary class $K$ of algebras the following are equivalent: \(^3\)

(a) $K$ is the reduct of some variety,

(b) $K$ can be axiomatized by strict Horn sentences,

(c) $K$ is closed under reduced products (including the empty product).

**Proof.** The equivalence of (b) and (c) is exercise 6.2.8 of [5]. If (a) holds then clearly (c) holds (we do not need the fact that $K$ is elementary for this implication). So suppose (b) holds, and let $\Sigma$ be a set of strict Horn sentences axiomatizing $K$.

\(^3\) Our proof of (b)$\Rightarrow$(a) is essentially that of T. Evans [6]. See McKenzie’s fascinating paper [12] for techniques based on the discriminator function.
By Skolemizing the sentences in $\Sigma$ we obtain $\Sigma^*$, a set of strict universal Horn sentences such that $K$ is a reduct of the models of $\Sigma^*$. Without loss of generality we can replace $\Sigma^*$ by a set $\Sigma^{**}$ of quasi-identities. Expand the language of $\Sigma^{**}$ by one new $(2n+1)$-ary function $f_n(x_1, y_1, \ldots, x_n, y_n, u)$ for each $n \geq 1$, and let $\Sigma^{***}$ be the set of identities

$$f_n(z_1, z_1, z_2, z_2, \ldots, z_n, z_n, u) = u \quad (n \geq 1)$$

plus all identities

$$f_n(p_1, q_1, \ldots, p_n, q_n, p) = f_n(p_1, q_1, \ldots, p_n, q_n, q)$$

where $p_1=q_1 \& \ldots \& p_n=q_n \to p=q$ is in $\Sigma^{**}$. Now one only needs to verify that reducts of algebras in $\Sigma^{***}$ obtained by dropping the $f_n$'s actually satisfy $\Sigma^{**}$ (which is rather evident), and that any model of $\Sigma^{**}$ can be expanded to a model of $\Sigma^{***}$ (say by defining $f_n(a_1, a_1, \ldots, a_n, a_n, b) = b$, and otherwise $f_n(a, \ldots) = a$). Thus $K$ is a reduct of the variety defined by $\Sigma^{***}$. (If $\Sigma$ were finite then we would only require finitely many $f_n$'s, hence $\Sigma^{***}$ would also be finite.)

An easy application of this general result is Grätzer's characterization of the spectra of varieties\(^3\). The spectrum of a variety is the set of cardinalities of the finite algebras in the variety.

**Corollary 4** (Grätzer [8]). Let $S$ be a subset of the natural numbers. Then $S$ is the spectrum of a variety iff $1 \in S$ and $S$ is closed under multiplication.

**Proof.** ($\Leftarrow$) Let $K$ be the class of sets which are either infinite or whose cardinality is in $S$. Then $K$ is an elementary class closed under reduced products, hence $K$ is a reduct of some variety $V$. But then $S$ is the spectrum of $V$. \(\square\)

\section*{§ 2. Quasi-Varieties}

A quasi-variety (see [11]) is an elementary class which can be axiomatized by quasi-identities (see § 1). If we start with a set $\Sigma$ of strict Horn sentences and Skolemize we (essentially) obtain a set $\Sigma^{**}$ of quasi-identities (this is of course the notation used in the proof of Theorem 3). In many cases $\Sigma^{**}$ already defines a variety – if so this gives a more natural variety of which the class defined by $\Sigma$ is a reduct. A non-trivial example of this is the class $K$ of pseudo-complemented semilattices with zero $(S, \wedge, 0)$ axiomatized by

\begin{align*}
(S1) \quad & x \wedge (y \wedge z) = (x \wedge y) \wedge z \\
(S2) \quad & x \wedge y = y \wedge x \\
(S3) \quad & x \wedge x = x \\
(S4) \quad & \forall x \exists y \exists z [x \wedge y = 0 \& (x \wedge z = 0 \to z \wedge y = z)].
\end{align*}

\(^3\) The usual proof is based on the study of primal clusters.
Skolemizing we obtain (S1)-(S3) plus

\[(S4') \quad x \wedge x^* = 0\]
\[(S5') \quad x \wedge z = 0 \rightarrow z \wedge x^* = z,\]
a quasi-variety which is indeed a variety. This is not an obvious fact; it was first pointed out in Balbes and Horn [2]. (See Sankappanavar [16] for a historical discussion.) We now look at the problem of determining when a quasi-variety is a variety. (One of the most interesting results in this direction is Ol'shanskii's [15] characterization of finite groups $G$ such that the quasi-variety generated by $G$ is a variety.)

**Theorem 5.** There is no algorithm to determine if a finite set of quasi-identities defines a variety.$^4$

**Proof.** For $\Pi$ as before let $\Sigma$ be $\Pi_{ab} \cup \{x+y=x \rightarrow x^* = y\}$. If $\Sigma \vdash x = y$ let $A$ be a model of $\Pi_{ab}$ which is not simple, and choose distinct elements $a_0, a_1$ of $A$ such that for some congruence $\theta$, $[a_0]_\theta = [a_1]_\theta$ and $A/\theta$ has at least two elements. Let $\Sigma \models A$ and define the operation $+$ on $A$ by $x+y = a_0$ if $x \leq y$, $x+y = a_1$ if $y < x$. The resulting algebra $A'$ is a model of $\Sigma$, and $\theta$ is a congruence of $A'$. As $A'/\theta \models x+y = y+x$ but $A'/\theta \not\models x = y$ it follows that $\Sigma$ does not define a variety. Consequently we see that $\Sigma$ will define a variety iff $\Sigma \models \neg x = y$, hence iff $\Pi_{ab} \models \neg x = y$, and again by Lemma 1 this is undecidable.

In spite of Theorem 5 we can still find some useful positive results. From basic facts in universal algebra (see [9]) we know that any quasi-variety $Q$ contains the free algebras of the variety generated by $Q$. If $\Sigma$ is a finite set of quasi-identities let $\Sigma$ be the subset of quasi-identities which are not already identities, and let $F(\Sigma)$ be the free algebra in the quasi-variety defined by $\Sigma$ which is freely generated by $n(\Sigma)$ elements, $n(\Sigma)$ being the maximum number of variables in a quasi-identity from $\Sigma$.

**Theorem 6.** There is an algorithm from which one can determine from any finite set $\Sigma$ of quasi-identities for which $F(\Sigma)$ is a finite algebra if $\Sigma$ determines a variety. This algorithm does not give an erroneous result when starting from a $\Sigma$ such that $F(\Sigma)$ is infinite, but maybe it does not terminate.

**Proof.** A universal sentence $\Phi$ with $n$ quantifiers will hold in a variety $V$ iff it holds in the $n$-generated algebras of the variety. Thus to see if $\Sigma$ defines a variety it suffices to construct $F(\Sigma)$ and see if all the homomorphic images of $F(\Sigma)$ satisfy $\Sigma$. If so then $\Sigma$ defines a variety, otherwise not. [We should point out that if we know $F(\Sigma)$ is finite, then, without even being given a finite bound on the size of $F(\Sigma)$, we can still construct $F(\Sigma)$ by enumerating the equational consequences $p(x_1, ..., x_n) = q(x_1, ..., x_n)$ of $\Sigma$, where $n = n(\Sigma)$, and checking

$^4$ Actually a deeper result is proved in McNulty [14] for universal Horn theories.
down the list until we find a collection of polynomials $p_1, \ldots, p_t$ containing $x_1, \ldots, x_n$ and such that, modulo $\Sigma$, it is closed under the fundamental operations. This will give a finite algebra $A$ which, after factoring by the smallest congruence $\theta$ such that $A/\theta \models \Sigma$, yields $F(\bar{\Sigma})$.

Let us apply this method to pseudo-complemented semilattices (with zero). Letting $\Sigma$ be (S1), (S2), (S3), (S4'), (S5') we have $n(\bar{\Sigma}) = 2$. As every meet semilattice $S$ with zero can be embedded in the distributive lattice with zero of non-empty lower segments of $S$ (a lower segment of $S$ is a subset $L$ such that $x \in S$, $y \in L$ imply $x \land y \in L$) we can apply the prime ideal theorem of Stone–Birkhoff (see [9]) to obtain, for any given $a, b \in S$ with $a \neq b$, a homomorphism of $S$ onto the two-element meet semilattice with zero. This is easily checked to preserve the Skolem operation $*$ of (S4'), (S5'). Hence the quasi-variety defined by $\Sigma$ is generated by a two-element algebra $A_0$. But then $F(\bar{\Sigma})$ must be a subalgebra of $A_0^4$, so from this point it is routine to find $F(\bar{\Sigma})$ and verify that all of its quotients satisfy $\bar{\Sigma}$.

One aspect of the previous paragraph does not appear to be constructive, namely knowing to use the Stone–Birkhoff result to show that $F(\bar{\Sigma})$ is finite. The next result shows that one must resort to special methods to show that a given $F(\bar{\Sigma})$ is finite.

**Theorem 7.** There is no algorithm by which one could determine from a finite set of quasi-identities $\Sigma$ whether or not $F(\bar{\Sigma})$ is finite.

**Proof.** Let $\Pi$ be as previously defined and let $\Sigma$ be the quasivariety defined by $\Pi \cup \{a = b \rightarrow x = y\}$, where we assume the language of $\Sigma$ has an additional binary operation $+$ (which does not appear in $\Sigma$). Then $F(\bar{\Sigma})$ is finite iff $\Pi \vdash a = b$, and this is undecidable.

§ 3. Homomorphism-Preserving Reducts of Varieties

Let us look at two examples of strict Horn classes where Skolemizing immediately yields a variety. **RCDL** is the class of relatively complemented distributive lattices $(L, \lor, \land)$. **RCDL** is axiomatized by the usual equations $\Sigma_D$ for distributive lattices plus the strict Horn sentence

$$\forall x \forall y \forall z \exists w[p(x, y, z) \lor w = x \lor z \quad \& \quad p(x, y, z) \land w = x \land z]$$

where $p(x, y, z) = (y \lor (x \land z)) \land (x \lor z)$. (This sentence says that, for $L$ a distributive lattice and elements $a, b, c \in L$, the element $b$ projected into the interval $[a \land c, a \lor c]$ has a relative complement in the interval.) Using a new ternary operation symbol $t$ we can Skolemize this sentence to obtain the variety $\Sigma^{**}$ defined by $\Sigma_D$ plus the equations $p(x, y, z) \lor t(x, y, z) = x \lor z$, $p(x, y, z) \land t(x, y, z) = x \land z$. (A closely related system is given in Balbes and Dwinger [1]; the one above is referred to in Burris and Werner [4].) The second example is the class **CBR** of commutative
bi-regular rings with unity \((R, +, \cdot, -, 0, 1)\) axiomatized by the usual commutative ring equations \(\Sigma_{CR}\) plus the strict Horn sentence \(\forall x \exists y \left[ y^2 = y \& x = xy \& x(1-y) = 0 \right]\) (which says that the principal ideal generated by \(x\) is also generated by an idempotent element \(y\)). Skolemizing we obtain a variety \(\text{CBR}^{**}\) of algebras \((R, +, \cdot, -\cdot, 0, 1, \cdot)\) axiomatized by \(\Sigma_{CR}\) plus \((x')^2 = x', \ x = xx', \text{ and } x(1-x') = 0\).

In both of these examples the variety created by Skolemizing the strict Horn sentence has the same homomorphisms as the original class. Before stating a general theorem which will prove this assertion let us look at a closely related example where Skolemizing destroys some of the homomorphisms. Let \(\Sigma_{BL}\) be a set of equations defining bounded lattices \((L, \lor, \land, 0, 1)\), and let \(\Sigma\) be \(\Sigma_{BL}\) plus the sentence \(\forall x \exists y (x \land y = 0 \& x \lor y = 1)\). Thus \(\Sigma\) axiomatizes the \(\text{(strict Horn)}\) class of complemented bounded lattices \(\text{CBL}\). Suppose \(\text{CBL}\) is a reduct of a variety \(V\) (not necessarily the obvious one obtained by Skolemizing). Then we will show that there are algebras \(A, B \in \text{CBL}\) such that for any expansions \(A', B'\) into \(V\) there will be a homomorphism \(\alpha: A \to B\) such that \(\alpha: A' \to B'\) is not a homomorphism. Namely let \(A\) be the four-element bounded lattice \(\{0, a, b, 1\}, \lor, \land, 0, 1\) with \(a \lor b = 1, \ a \land b = 0\), and let \(B\) be the non-modular five-element bounded lattice \(\{0, a, b_1, b_2, 1\}, \lor, \land, 0, 1\) where \(b_1 < b_2\). Then \(\alpha_1: A \to B\) defined by \(\alpha_1(0) = 0, \ \alpha_1(1) = 1, \ \alpha_1(a) = a\) and \(\alpha_1(b) = b_1\) is an embedding, as well as \(\alpha_2: A \to B\) defined by \(\alpha_2(x) = \alpha_1(x)\) if \(x \neq b\), \(\alpha_2(b) = b_2\). As \(\{0, a, 1\}\) is not a subuniverse of \(A'\) (note that \((\{0, a, 1\}, \lor, \land, 0, 1) \notin \text{CBL}\) it follows that there is a polynomial \(p(x, y, z)\) such that \(A' \models p(0, a, 1) = b\). Now \(p(a_0, a_a, a_1) = p(a_0, a_2, a_2)\), but \(a_1 b \neq a_2 b\), hence either \(\alpha_1: A' \to B'\) or \(\alpha_2: A' \to B'\) is not a homomorphism. With this example in mind perhaps the hypotheses of the following theorem will seem rather natural. The symbol \(\exists!\) denotes "there exists a unique".

**Theorem 8.** Let \(K\) be a class of algebras defined by a set \(\Sigma\) of positive first-order sentences in prefix form, i.e. by sentences of the form \(\forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \Phi(\vec{x}, \vec{y})\), where \(\Phi(\vec{x}, \vec{y})\) is positive and quantifier-free. Let \(U\) be the universal class defined by the set \(\Sigma^*\) of universal sentences obtained by Skolemizing \(\Sigma\). Then the following are equivalent:

(i) every algebra in \(K\) is the reduct of a unique algebra in \(U\);

(ii) if \(A, B \in K\) and \(\alpha: A \to B\) is a homomorphism then \(\alpha: A' \to B'\), where \(A', B'\) are expansions of \(A\) into \(U\), is also a homomorphism;

(iii) for each sentence \(\forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \Phi(\vec{x}, \vec{y})\) in \(\Sigma\) we have \(\Sigma \models \forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \Phi(\vec{x}, \vec{y})\).

**Proof.** (ii) \(\Rightarrow\) (iii). Suppose that for some sentence \(\forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \Phi(\vec{x}, \vec{y})\) in \(\Sigma\) we have \(\Sigma \models \forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \Phi(\vec{x}, \vec{y})\). Then there is a model \(A\) of \(\Sigma\) such that \(A \models \forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \Phi(\vec{x}, \vec{y})\), hence there are two distinct choices \(\vec{f}\) and \(\vec{g}\) of Skolem functions of \(\forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \Phi(\vec{x}, \vec{y})\) for \(A\). Thus \(A' \models \forall \vec{x} \Phi(\vec{x}, \vec{f}(\vec{x}))\) and \(A'' \models \forall \vec{x} \Phi(\vec{x}, \vec{g}(\vec{x}))\), where \(A'\) is the expansion of \(A\) by \(\vec{f}\) and \(A''\) is the expansion of \(A\) by \(\vec{g}\). Since \(f_i \neq g_i\) for some \(i\), choose \(\vec{a}\) from
$A$ such that $f_i(\bar{a}) \neq g_i(\bar{a})$. Then with $\alpha$ the identity map on $A$ we have $\alpha f_i(\bar{a}) \neq \alpha g_i(\bar{a})$, so $\alpha: A' \to A''$ is not a homomorphism, so (ii) fails if (iii) fails.

(i) $\Rightarrow$ (iii). Use the $A'$ and $A''$ above to show (i) fails if (iii) fails.

(iii) $\Rightarrow$ (i). If (iii) holds then the Skolem functions for each model of $\Sigma$ are unique for each $A \in K$, so (i) holds.

Remark. Conditions (i) and (ii) together state that the forgetful functor from $\mathbf{U}$ to $\mathbf{K}$ is an isomorphism.

Remark. In Theorem 8 the class $\mathbf{U}$ is a variety if the sentences in $\Sigma$ are of the form $\forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \Phi(x, y) \in \Sigma$ and let $f_A$, respectively $f_B$, be the unique Skolemizing functions of this sentence on $A$, respectively $B$. For $a_1, \ldots, a_n \in A$, $A \models \Phi(\bar{a}, f_A(\bar{a}))$, hence $B \models \Phi(\bar{a}, f_B(\bar{a}))$ as positive formulas are preserved by homomorphisms. But $B \models \Phi(\bar{a}, f_B(\bar{a}))$, so from the uniqueness properties, $\alpha f_A(\bar{a}) = f_B(\bar{a})$, hence $\alpha$ preserves the Skolem functions.

It is easy to prove that our axiom system for RCDL has the desired uniqueness properties. Hence the only subdirectly irreducible member of RCDL is the two-element algebra $(2, \lor, \land, t)$, where $2 = \{0, 1\}$, and this algebra satisfies

$$t(x, y, z) = \begin{cases} x & \text{if } x \neq y \\ z & \text{if } x = y. \end{cases}$$

This defines the remarkable ternary discriminator function which, by the theorem of Keimel and Werner (see [4]), guarantees that every algebra in RCDL is isomorphic to a Boolean product (see [4]) of one- and two-element algebras. Results in Burris [3] furthermore say that every algebra in RCDL is isomorphic to the algebra of all continuous functions $f$ from a Boolean space $Y$ to $(2, \lor, \land, t)$ such that for some fixed pair of elements $x_i \in X$, $f(x_i) = i$, $i = 0, 1$. (This gives Feinberg’s [7] characterization of members of RCDL as the sublattices of Boolean lattices obtained by intersecting one maximal ideal with one maximal filter.)

Likewise we can apply Theorem 8 to the axioms for CBR to show that the reducts of subdirectly irreducible members of CBR are precisely the fields. Hence for $R'$ a subdirectly irreducible member of CBR the function $(x - y)^t \cdot x + [1 - (x - y)] \cdot z$ is the ternary discriminator on $R'$, so applying the Bulman-Fleming and Werner (see [4]) results on discriminator varieties it follows that every commutative biregular ring is a Boolean product of fields.

Acknowledgements. We take this opportunity to thank Professor Bulman-Fleming for interesting discussions on the topic of this paper, and the referee for improving the formulation of some of the results and the sharpened version of Theorem 8 presented here.
References


Department of Pure Mathematics, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, Canada, N2L 3G1

*Received November 16, 1977; revised April 7, 1978*