EMBEDDING THE DUAL OF $\Pi_n$ IN THE LATTICE OF EQUATIONAL CLASSES OF COMMUTATIVE SEMIGROUPS

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ABSTRACT. The lattice of equational classes of commutative semigroups does not satisfy any special lattice laws.

In [4] R. Schwabauer proved that the lattice $\mathcal{L}$ of equational classes of commutative semigroups is nonmodular. In this paper we will prove a maximal extension of this result, namely, $\mathcal{L}$ does not satisfy any special lattice laws.

The free commutative semigroup on countably many generators, $\mathfrak{F}(\omega)$, is the set of sequences $(u_n)_{n \in \mathbb{N}}$ of nonnegative integers, such that $u_n = 0$ for all but finitely many $n \in \mathbb{N}$ and $\sum u_n \geq 1$, with component-wise addition. [For convenience, we write $(u_n)$ for $(u_n)_{n \in \mathbb{N}}$.]

A commutative semigroup equation is a pair $((u_n), (v_n))$ of elements of $\mathfrak{F}(\omega)$ (see [1]). A commutative semigroup $(S, \cdot)$ satisfies the equation $((u_n), (v_n))$ iff, for every family $(a_n)$ of elements of $S$,

$\Pi \{ a_n^m \mid u_n \neq 0 \} = \Pi \{ a_n^m \mid v_n \neq 0 \}$.

A set $\Sigma$ of equations is closed [1, p. 170, Definition 2] iff it contains all trivial equations, is symmetric and transitive, and is closed under multiplication and substitution of terms for variables. Thus $\Sigma$ is closed iff it satisfies the following conditions:

(P1): $((u_n), (u_n)) \in \Sigma$ for all $(u_n) \in \mathfrak{F}(\omega)$.

(P2): If $((u_n), (v_n)) \in \Sigma$ then $((v_n), (u_n)) \in \Sigma$.

(P3): If $((u_n), (v_n)) \in \Sigma$ and $((v_n), (w_n)) \in \Sigma$ then $((u_n), (w_n)) \in \Sigma$.

(P4): If $((u_n), (v_n)) \in \Sigma$ and $((u_n'), (v_n')) \in \Sigma$ then $((u_n + u_n'), (v_n + v_n')) \in \Sigma$.

(P5): If $((u_n), (v_n)) \in \Sigma$, $(k_n) \in \mathfrak{F}(\omega)$ and $p \in \mathbb{N}$, then the result of "substituting $(k_n)$ for the $p$th variable" in $((u_n), (v_n))$ is in $\Sigma$, i.e., $((w_n), (x_n)) \in \Sigma$ where $w_n = u_n + k_n u_p$ for $n \neq p$, $w_p = k_p u_p$, and $x_n = v_n + k_n v_p$ for $n \neq p$ and $x_p = k_p v_p$.

(Note that these conditions (P1) to (P5) are, essentially, a restatement of conditions (i) to (v) in Grätzer [1, p. 170, Definition 2]; in condition (iv) we need only consider the one binary operation, hence

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the different form of (P4). \( \Gamma(\Sigma) \) will denote the deductive closure of \( \Sigma \).

Let \( \mathcal{L}' \) be the lattice of closed sets of commutative semigroup equations; then \( \mathcal{L}' \) is dually isomorphic to the lattice \( \mathcal{L} \) of equational classes of commutative semigroups.

For each \( m \in \mathbb{N} \), let \( \Pi_m \) be the partition lattice on \( \{1, 2, \cdots, m\} \).

**Theorem 1.** For each \( m \in \mathbb{N} \), \( \Pi_m \) is (isomorphic to) a sublattice of \( \mathcal{L}' \).

**Proof.** Let \( m \) be a fixed natural number. If \( \pi \in \Pi_m \), we write \( \equiv \) for the equivalence relation on \( \{1, 2, \cdots, m\} \) induced by \( \pi \). Let 
\[
\Sigma = \{ ((u_n), (v_n)) \mid \sum u_n, \sum v_n \geq 2m + 2 \} \cup \{ ((u_n), (u_n)) \mid (u_n) \in \mathcal{F}(\omega) \}.
\]

It is clear from (P1) to (P5) that \( \Sigma \) is a closed set of equations. For each \( \pi \in \Pi_m \), define a set \( \Sigma(\pi) \) of equations as follows: \( ((u_n), (v_n)) \in \Sigma(\pi) \) iff there exist \( j, k \) with \( u_n = v_n = 0 \) for all \( n \neq j, k \), \( u_j + u_k = 2m + 1 \) = \( v_j + v_k \), and either \( u_j = v_j \) or \( u_k = v_k \). Note that if \( u_j + u_k = 2m + 1 \) and \( u_j = v_j \), then \( u_k \neq 0 \). This is not equivalent to anything modulo \( \pi \). Then, since \( \pi \) is a partition, it follows that if \( ((u_n), (v_n)) \in \Sigma(\pi) \) and if \( ((v_n), (w_n)) \in \Sigma(\pi) \), then \( ((u_n), (w_n)) \in \Sigma(\pi) \). \( \Sigma(\pi) \) is obviously symmetric. Applying (P4) to two equations in \( \Sigma(\pi) \) yields an equation in \( \Sigma \). Applying (P5) with \( \sum k_n = 1 \) to an equation in \( \Sigma(\pi) \) yields either a trivial equation or an equation in \( \Sigma(\pi) \); applying (P5) with \( \sum k_n \geq 2 \) and with \( u_p \geq 1 \) yields an equation in \( \Sigma \); and applying (P5) with \( u_p = 0 \) does not change the equation. Thus \( \Sigma \cup \Sigma(\pi) \) is a closed set of equations.

For two partitions \( \pi_1, \pi_2 \), if \( \pi_1 \wedge \pi_2 \) and \( \pi_1 \vee \pi_2 \) are the meet and join of \( \pi_1 \) and \( \pi_2 \) in \( \Pi_m \), then 
\[
(\Sigma \cup \Sigma(\pi_1)) \cap (\Sigma \cup \Sigma(\pi_2)) = \Sigma \cup (\Sigma(\pi_1) \cap \Sigma(\pi_2)) = \Sigma \cup (\Sigma(\pi_1 \wedge \pi_2)).
\]

Also it is clear that 
\[
(\Sigma \cup \Sigma(\pi_1)) \cup \mathcal{L}'(\Sigma \cup \Sigma(\pi_2)) = \Gamma(\Sigma \cup \Sigma(\pi_1) \cup \Sigma(\pi_2))
\]
\[
\subseteq \Sigma \cup \Sigma(\pi_1 \wedge \pi_2).
\]

Conversely, if \( ((u_n), (v_n)) \in \Sigma(\pi_1 \wedge \pi_2) \), then there exist \( j, k \) with \( u_n = v_n = 0 \) for all \( n \neq j, k \), \( u_j + u_k = v_j + v_k = 2m + 1 \), and, w.l.o.g., \( u_j \equiv \pi_1 v_j \). But then there exist \( w_1, \cdots, w_p \) in \( \{1, 2, \cdots, m\} \) such that \( w_1 = u_i, w_p = v_i \), and \( w_i \equiv \pi_i w_{i+1} \) for \( i \) odd, \( w_i \equiv \pi_i w_{i+1} \) for \( i \) even. Let \( \alpha_i \in \mathcal{F}(\omega) \) have \( f \)th entry \( w_i, k \)th entry \( 2m + 1 - w_i \), and all other entries zero. Then \( \alpha_1 = (u_n), \alpha_p = (v_n) \) and \( (\alpha_i, \alpha_{i+1}) \in \Sigma(\pi_i) \) for \( i \) odd and \( (\alpha_{i}, \alpha_{i+1}) \in \Sigma(\pi_2) \) for \( i \) even. It follows that 
\[
((u_n), (v_n)) = (\alpha_1, \alpha_p) \in \Gamma(\Sigma(\pi_1) \cup \Sigma(\pi_2)).
\]

Hence
\[ \Sigma \cup \Sigma(\pi_1 \lor \pi_2) \subseteq \Gamma(\Sigma(\pi_1) \cup \Sigma(\pi_2) \cup \Sigma). \]

Thus we have
\[ (\Sigma \cup \Sigma(\pi_1)) \land \mathcal{L} \cdot (\Sigma \cup \Sigma(\pi_2)) = \Sigma \cup \Sigma(\pi_1 \land \pi_2) \]
and
\[ (\Sigma \cup \Sigma(\pi_1)) \lor \mathcal{L} \cdot (\Sigma \cup \Sigma(\pi_2)) = \Sigma \cup \Sigma(\pi_1 \lor \pi_2). \]

It follows that the mapping \( \pi \rightarrow \Sigma \cup \Sigma(\pi) \) is a homomorphism of \( \Pi_m \)
into \( \mathcal{L} \).

It is clear that if \( \pi_1 \neq \pi_2 \) then \( \Sigma \cup \Sigma(\pi_1) \neq \Sigma \cup \Sigma(\pi_2) \); thus this homomorphism is one-to-one, and this yields the desired result.

**Theorem 2.** \( \mathcal{L} \) does not satisfy any special lattice laws.

**Proof.** From D. Sachs [3] it is known that the family of partition lattices \( \Pi_m, m = 1, 2, \cdots \), does not satisfy any special lattice laws.

**Concluding Remark.** One might consider the possibility of embedding the dual of \( \Pi_m \) into \( \mathcal{L} \), but a recent paper of P. Perkins [2] shows that this is impossible because \( \mathcal{L} \) is countable, whereas \( \Pi_m \) is uncountable.

**References**


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