Reprint from
algebra universalis
Vol. 1, fasc. 1, 1971 BIRKHÄUSER VERLAG BASEL pages 39–45

On the Structure of the Lattice of Equational Classes $\mathscr{L}(\tau)$

STANLEY BURRIS[1])

Introduction

For a given similarity type τ, the number of atoms of $\mathscr{L}(\tau)$ will be determined and the possibility of embedding partition lattices into $\mathscr{L}(\tau)$ will be discussed. Our result gives a complete solution to Problem 33 of Grätzer (see [2], page 194), and the discussion gives some insight into the difficulties of Grätzer's Problem 32 (the problem of characterizing $\mathscr{L}(\tau)$ lattice-theoretically).

The notation used is that of Grätzer [2]. A *type* τ of algebras is a sequence $\langle n_0, n_1, \ldots, n_\gamma, \ldots \rangle$ of non-negative integers n_γ, for $\gamma < 0(\tau)$, where $0(\tau)$ is an ordinal. For every $\gamma < 0(\tau)$ one has an *n_γ-ary operation symbol* $\mathbf{f}_\gamma$. The set $\mathbf{P}^{(n)}(\tau)$ of *n-ary polynomial symbols* of type τ is defined as follows:

1.1. $\mathbf{x}_0, \ldots, \mathbf{x}_{n-1}$ are n-ary polynomial symbols;

1.2. if $\mathbf{p}_0, \ldots, \mathbf{p}_{n_\gamma - 1}$ are n-ary polynomial symbols and $\gamma < 0(\tau)$, then $\mathbf{f}_\gamma(\mathbf{p}_0, \ldots$ $\ldots, \mathbf{p}_{n_\gamma - 1})$ is an n-ary polynomial symbol;

1.3. n-ary polynomial symbols are precisely those obtained from 1.1 and 1.2 in a finite number of steps.

Define $\mathbf{P}^{(\omega)}(\tau) = \bigcup_{n<\omega} \mathbf{P}^{(n)}(\tau)$. A *$\tau$-identity* is an expression of the form $\mathbf{p} = \mathbf{q}$ where $\mathbf{p}, \mathbf{q} \in \mathbf{P}^{(\omega)}(\tau)$.

Let $\mathscr{I}$ denote the set of all τ-identities and define the operators $\mathfrak{R}$, $\mathfrak{S}$ and $\mathfrak{T}$ on $2^{\mathscr{I}}$ by:

$$\mathfrak{R}(\mathscr{I}) = \{\mathbf{x}_0 = \mathbf{x}_0\}$$
$$\mathfrak{S}(\mathscr{I}) = \{\mathbf{p} = \mathbf{q} : \mathbf{p} = \mathbf{q} \in \mathscr{I} \quad \text{or} \quad \mathbf{q} = \mathbf{p} \in \mathscr{I}\}$$
$$\mathfrak{T}(\mathscr{I}) = \{\mathbf{p} = \mathbf{q} : \mathbf{p} = \mathbf{r} \in \mathscr{I} \quad \text{and} \quad \mathbf{r} = \mathbf{q} \in \mathscr{I} \text{ for some } \mathbf{r} \in \mathbf{P}^{(\omega)}(\tau)\}.$$

A set Σ of identities is *closed* provided:

2.1. $\mathfrak{R}(\Sigma) \subset \Sigma$;

2.2. $\mathfrak{S}(\Sigma) \subset \Sigma$;

2.3. $\mathfrak{T}(\Sigma) \subset \Sigma$;

2.4. if $\mathbf{p}_i = \mathbf{q}_i$ is in Σ for $i = 0, \ldots, n_\gamma - 1$, then so is $\mathbf{f}_\gamma(\mathbf{p}_0, \ldots, \mathbf{p}_{n_\gamma - 1}) = \mathbf{f}_\gamma(\mathbf{q}_0, \ldots, \mathbf{q}_{n_\gamma - 1})$;

2.5. if $\mathbf{p} = \mathbf{q}$ is in Σ, and $\mathbf{p}' = \mathbf{q}'$ is derived from $\mathbf{p} = \mathbf{q}$ by replacing all occurrences of $\mathbf{x}_i$ by an arbitrary polynomial symbol $\mathbf{r}$, then $\mathbf{p}' = \mathbf{q}'$ is in Σ.

The smallest closed set of identities containing Σ will be denoted by $[\Sigma]$.

First, three well-known results will be stated without proof (see [2] and [7]).

[1]) Research supported by NRC Grant A7256.

Presented by R. McKenzie. Received April 23, 1970. Accepted for publication September 20, 1970.

THEOREM 1: *The closed sets of τ-identities constitute a complete lattice $\mathscr{L}(\tau)$ under the partial ordering: $\Sigma_1 \leqslant \Sigma_2$ if $\Sigma_1 \supset \Sigma_2$.*

THEOREM 2: *Every point of $\mathscr{L}(\tau)$ is above an atom (equationally complete set of identities) of $\mathscr{L}(\tau)$.*

In several of the following results we require a detailed approach to constructing $[\Sigma]$ from Σ (rather than merely assuring the existence of $[\Sigma]$ by intersecting the closed sets of identities which contain Σ). Before stating Theorem 3, we briefly introduce the following notation to explain the syntactic approach. $Su(\tau)$ will denote the set of substitutions for the polynomials $\mathbf{P}^{(\omega)}(\tau)$ (more precisely, $Su(\tau)$ is the set of endomorphisms of the absolutely free algebra of type τ on the countable set of generators $\mathbf{x}_0, \mathbf{x}_1, \ldots$), and for $\mathbf{r} \in \mathbf{P}^{(\omega)}(\tau)$, $S_{\mathbf{r}}^{\mathbf{x}_0}$ will denote the substitution which replaces $\mathbf{x}_0$ by $\mathbf{r}$ and leaves $\mathbf{x}_i$ fixed for $i > 0$. Let $Su^*(\tau)$ be the set of substitutions which map variable symbols to variable symbols. Also, let $\overline{\mathfrak{T}} = \mathfrak{T} \cup \mathfrak{T}^2 \cup \ldots$.

THEOREM 3: *Let Σ be a set of τ-identities. Then*

$$[\Sigma] = \overline{\mathfrak{T}}\left(\bigcup \{[\mathbf{p} = \mathbf{q}] : \mathbf{p} = \mathbf{q} \in \Sigma\}\right)$$

where

$$[\mathbf{p} = \mathbf{q}] = \overline{\mathfrak{T}}\mathfrak{S}\left(\{\sigma S_{\varrho(\mathbf{p})}^{\mathbf{x}_0}(\mathbf{r}) = \sigma S_{\varrho(\mathbf{q})}^{\mathbf{x}_0}(\mathbf{r}) : \sigma \in Su^*(\tau), \varrho \in Su(\tau), \mathbf{r} \in \mathbf{P}^{(\omega)}(\tau)\}\right).$$

When it is appropriate to emphasize that $\mathbf{p} = \mathbf{q} \in [\Sigma]$ can be derived using Theorem 3, it is usual to employ the notation $\Sigma \vdash \mathbf{p} = \mathbf{q}$.

If τ is a similarity type, the *multiplicity type* μ associated with τ is $\mu = \langle \mathfrak{m}_0, \ldots \ldots, \mathfrak{m}_i, \ldots \rangle_{i < \omega}$ where

$$\mathfrak{m}_i = |\{\gamma : \gamma < 0(\tau) \quad \text{and} \quad n_\gamma = i\}|.$$

I. The Atoms of $\mathscr{L}(\tau)$.

There are two classical results on $\mathscr{L}(\tau)$ which were proved in the early fifties (see [5] and [6]).

THEOREM 4: (Kalicki and Scott) *$\mathscr{L}(\langle 1 \rangle)$ has two atoms.*

THEOREM 5: (Kalicki) *$\mathscr{L}(\langle 2 \rangle)$ has $2^{\aleph_0}$ atoms.*

The remaining cases will be examined in this section.

THEOREM 6: *Let $\tau = \langle n_0, n_1, \ldots, n_\gamma, \ldots \rangle$ be a similarity type and μ be the associated multiplicity type. Define $A(\tau)$ to be the number of atoms of $\mathscr{L}(\tau)$. The following table relates the similarity type τ to the number of atoms $A(\tau)$:*

	$\mathfrak{m}_2 = \mathfrak{m}_3 = \cdots \mathfrak{m}_i = \cdots = 0$			$\mathfrak{m}_j \neq 0$ for some $j \geqslant 2$
	$\mathfrak{m}_1 = 0$	$\mathfrak{m}_1 = 1$	$\mathfrak{m}_1 \geqslant 2$	
$A(\tau)$	1	2	$\operatorname{Max}(2^{\mathfrak{m}_1}, 2^{\aleph_0})$	$\operatorname{Max}(2^{\Sigma_{i<\omega}\mathfrak{m}_i}, 2^{\aleph_0})$

Proof: Since one can only form $\max(2^{\Sigma_{i<\omega}\mathfrak{m}_i}, 2^{\aleph_0})$ sets of τ-identities then $A(\tau) \leqslant \max(2^{\Sigma_{i<\omega}\mathfrak{m}_i}, 2^{\aleph_0})$. The following cases will correspond to the columns of the table.

(1) $\mathfrak{m}_1 = \mathfrak{m}_2 = \cdots = 0$. The only equationally complete set of identities is easily seen to be $[\{\mathbf{f}_\alpha = \mathbf{f}_\beta : \alpha, \beta < 0(\tau)\}]$.

(2) $\mathfrak{m}_1 = 1$, $\mathfrak{m}_2 = \mathfrak{m}_3 = \cdots = 0$. One can assume without loss of generality that $n_0 = 1$. The two systems

(*) $[\{\mathbf{f}_0(\mathbf{x}_0) = \mathbf{x}_0\} \cup \{\mathbf{f}_\alpha = \mathbf{f}_\beta : 1 \leqslant \alpha, \beta < 0(\tau)\}]$ and

(**) $[\{\mathbf{f}_0(\mathbf{x}_0) = \mathbf{f}_0(\mathbf{x}_1)\} \cup \{\mathbf{f}_\alpha = \mathbf{f}_0(\mathbf{x}_0): \quad 1 \leqslant \alpha < 0(\tau)\}]$

are readily seen to be distinct atoms, and any consistent system Σ is contained in one of these two – if Σ has an identity of the form $\mathbf{f}_0^n(\mathbf{x}_\nu) = \mathbf{x}_\nu$ then Σ is contained in (*), otherwise in (**).

(3) $\mathfrak{m}_1 \geqslant 2$, $\mathfrak{m}_2 = \mathfrak{m}_3 = \cdots = 0$.

(a) To guarantee the existence of at least $2^{\aleph_0}$ atoms, consider $\mathscr{L}(\langle 1, 1\rangle)$. Let N^+ be the positive integers, and let A and B be subsets of N^+. Define $\Sigma(A, B)$ to be the set

$$\{\mathbf{f}_0\mathbf{f}_1\mathbf{f}_0^n\mathbf{f}_1^2(\mathbf{x}_0) = \mathbf{x}_0 : n \in A\} \cup \{\mathbf{f}_0\mathbf{f}_1\mathbf{f}_0^n\mathbf{f}_1^2(\mathbf{x}_0) = \mathbf{f}_0\mathbf{f}_1\mathbf{f}_0^n\mathbf{f}_1^2(\mathbf{x}_1) : n \in B\}$$

If $A \cap B \neq \emptyset$, then $\Sigma(A, B)$ is inconsistent since a function cannot be both constant and the identity except on the trivial algebra.

If $A \cap B = \emptyset$, then the consistency of $\Sigma(A, B)$ is seen from the following model. Define f_0 and f_1 on N^+ by $f_1(n) = 2^n$, $f_0(2^{2^n}) = 3^n$, $f_0(p_{m+1}^n) = p_{m+2}^n$, where $m \geqslant 1$ and p_m is the mth prime, and $f_0(2^{p_{m+1}^n}) = n$ if $m \in A$, and $f_0(t) = 1$ otherwise. This is readily checked to be a model of $\Sigma(A, B)$.

By considering distinct complementary pairs (A, B) and (A_1, B_1), one has in $\mathscr{L}(\langle 1, 1\rangle)$ that $[\Sigma(A, B)] \wedge [\Sigma(A_1, B_1)] = 0$, since either $A_1 \cap B \neq \emptyset$ or $A \cap B_1 \neq \emptyset$. From Theorem 2 it follows that $\mathscr{L}(\langle 1, 1\rangle)$ has a continuum of atoms.[2])

(b) To show that there are at least $2^{\mathfrak{m}_1}$ atoms, let $J = \{\gamma < 0(\tau) : n_\gamma = 1\}$. For each partition (A, B) of J into two sets, let

$$\Sigma(A, B) = \{\mathbf{f}_\gamma(\mathbf{x}_0) = \mathbf{x}_0 : \gamma \in A\} \cup \{\mathbf{f}_\gamma(\mathbf{x}_0) = \mathbf{f}_\gamma(\mathbf{x}_1) : \gamma \in B\}.$$

2) G. McNulty pointed out to the author that the result of paragraph (a) has also been obtained by F. Backer and R. Thompson.

These sets are consistent, hence one has $A(\tau) \geqslant 2^{\mathfrak{m}_1}$ by an argument similar to that of paragraph (a). Combining (a) and (b), one has $A(\tau) \geqslant \max(2^{\mathfrak{m}_1}, 2^{\aleph_0})$.

(c) Finally, to prove that there are no more than $\max(2^{\mathfrak{m}_1}, 2^{\aleph_0})$ atoms, let $J = \{\gamma : n_\gamma = 1\}$ and $K = \{\gamma : n_\gamma = 0\}$, and suppose that Σ is a consistent set of τ-identities. The basic idea is to show that

$$\Sigma^+ = \Sigma \cup \{\mathbf{f}_\alpha = \mathbf{f}_\beta : \alpha, \beta \in K\} \cup \{\mathbf{f}_\lambda(\mathbf{f}_\alpha) = \mathbf{f}_\alpha : \alpha \in K, \lambda \in J\}$$

is still consistent, and hence the atoms of $\mathscr{L}(\tau)$ must identity all polynomials involving nullary operations. Thus there will be no more equationally complete sets of τ-identities than there are sets of identities in the unary and variable symbols; that is, $A(\tau) \leqslant \max(2^{\mathfrak{m}_1}, 2^{\aleph_0})$.

Assume that Σ is a consistent set of identities. If $\Sigma^+ \vdash \mathbf{x}_0 = \mathbf{x}_1$, then by Theorem 3 there is a sequence of identities

$$\mathbf{x}_0 = \mathbf{p}_0, \mathbf{p}_0 = \mathbf{p}_1, \ldots, \mathbf{p}_{n-1} = \mathbf{p}_n, \mathbf{p}_n = \mathbf{x}_1$$

such that each identity is in some $[\mathbf{p} = \mathbf{q}]$ where $\mathbf{p} = \mathbf{q} \in \Sigma^+$. One can assume that the sequence above is minimal, and thus none of the identities are a consequence of $\mathbf{x}_0 = \mathbf{x}_0$. Clearly $\mathbf{x}_0 = \mathbf{p}_0 \notin [\mathbf{p} = \mathbf{q}]$ for any $\mathbf{p} = \mathbf{q} \in \Sigma^+ - \Sigma$. Let $\mathbf{p}_{k-1} = \mathbf{p}_k$ be the first identity such that $\mathbf{p}_{k-1} = \mathbf{p}_k \in [\mathbf{p} = \mathbf{q}]$ for some $\mathbf{p} = \mathbf{q} \in \Sigma^+ - \Sigma$. But then $\mathbf{p}_{k-1}$ does not involve a variable symbol (only unary and nullary symbols), and $\Sigma \vdash \mathbf{x}_0 = \mathbf{p}_{k-1}$, and hence $\Sigma \vdash \mathbf{x}_1 = \mathbf{p}_{k-1}$ (by uniform substitution), so $\Sigma \vdash \mathbf{x}_0 = \mathbf{x}_1$. This contradicts the consistency of Σ, and completes the proof of our theorem in case (3).

(4) $\mathfrak{m}_j \neq 0$ for some $j \geqslant 2$. This will be divided into three cases.

(i) $\Sigma_{i<\omega} \mathfrak{m}_i$ infinite, $\mathfrak{m}_0 \leqslant \Sigma_{i \geqslant 1} \mathfrak{m}_i$. Then $\Sigma_{i \geqslant 1} \mathfrak{m}_i = \Sigma_{i<\omega} \mathfrak{m}_i$ and one can obviously argue by analogy with paragraph (b) to obtain the desired conclusion.

(ii) $\Sigma_{i<\omega} \mathfrak{m}_i < \aleph_0$. Let $\mathbf{f}_\alpha$ be a polynomial symbol such that $n_\alpha \geqslant 2$. One can define a binary symbol $+$ by $\mathbf{x}_0 + \mathbf{x}_1 = \mathbf{f}_\alpha(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_1, \ldots, \mathbf{x}_1)$, and then using Kalicki's equations [5], one can extend $\mathbf{f}_\alpha$ arbitrarily in the Kalicki models to show that the equations are consistent, (where all other functions may be defined arbitrarily) and hence there are $2^{\aleph_0}$ atoms – clearly $2^{\Sigma_{i<\omega} \mathfrak{m}_i} < 2^{\aleph_0}$.

(iii) $\mathfrak{m}_0 > \Sigma_{i \geqslant 1} \mathfrak{m}_i$, $\mathfrak{m}_0 \geqslant \aleph_0$. Then $\Sigma_{i<\omega} \mathfrak{m}_i = \mathfrak{m}_0$. To get $2^{\mathfrak{m}_0}$ distinct atoms in $\mathscr{L}(\tau)$, for each partition (A, B) of $\{\gamma : n_\gamma = 0\}$ put

$$\Sigma(A, B) = \{\mathbf{f}_\gamma + \mathbf{x}_0 = \mathbf{x}_0 : \gamma \in A\} \cup \{\mathbf{f}_\gamma + \mathbf{x}_0 = \mathbf{f}_\gamma + \mathbf{x}_1 : \gamma \in B\},$$

where '+' is defined using $\mathbf{f}_\alpha$ as in (ii). Clearly, each set $\Sigma(A, B)$ is consistent. Hence the result follows immediately.

II. Embedding the Duals of Partition Lattices in $\mathscr{L}(\tau)$.

As before, μ will be the multiplicity type associated with τ.

THEOREM 7: *$\mathscr{L}(\langle 1, 1\rangle)$ contains the dual of the partition lattice of a countable set as a complete sublattice.*

Proof: Let Π_ω denote the partition lattice of N^+. To each $\pi \in \Pi_\omega$ associate a set of laws:

$$\Sigma(\pi) = \{\mathbf{f}_0\mathbf{f}_1^n\mathbf{f}_0(\mathbf{x}_0) = \mathbf{f}_0\mathbf{f}_1^m\mathbf{f}_0(\mathbf{x}_0) \colon m, n \in A \quad \text{for some} \quad A \in \pi\}.$$

The only laws which can be derived from $\Sigma(\pi)$ are of one of the following four forms:

(i) $\mathbf{f}_0^{s_1}\mathbf{f}_1^{s_2}\mathbf{f}_0^{s_3}\ldots\mathbf{f}_0^{s_{n-2}}\mathbf{f}_1^{s_{n-1}}\mathbf{f}_0^{s_n}(\mathbf{x}_\nu) = \mathbf{f}_0^{s_1}\mathbf{f}_1^{t_2}\mathbf{f}_0^{s_3}\ldots\mathbf{f}_0^{s_{n-2}}\mathbf{f}_1^{t_{n-1}}\mathbf{f}_0^{s_n}(\mathbf{x}_\nu)$;

(ii) $\mathbf{f}_0^{s_1}\mathbf{f}_1^{s_2}\mathbf{f}_0^{s_3}\ldots\mathbf{f}_1^{s_{n-3}}\mathbf{f}_0^{s_{n-2}}\mathbf{f}_1^{s_{n-1}}(\mathbf{x}_\nu) = \mathbf{f}_0^{s_1}\mathbf{f}_1^{t_2}\mathbf{f}_0^{s_3}\ldots\mathbf{f}_1^{t_{n-3}}\mathbf{f}_0^{s_{n-2}}\mathbf{f}_1^{s_{n-1}}(\mathbf{x}_\nu)$

(iii) $\mathbf{f}_1^{s_2}\mathbf{f}_0^{s_3}\mathbf{f}_1^{s_4}\ldots\mathbf{f}_0^{s_{n-2}}\mathbf{f}_1^{s_{n-1}}\mathbf{f}_0^{s_n}(\mathbf{x}_\nu) = \mathbf{f}_1^{s_2}\mathbf{f}_0^{s_3}\mathbf{f}_1^{t_4}\ldots\mathbf{f}_0^{s_{n-2}}\mathbf{f}_1^{t_{n-1}}\mathbf{f}_0^{s_n}(\mathbf{x}_\nu)$;

(iv) $\mathbf{f}_1^{s_2}\mathbf{f}_0^{s_3}\mathbf{f}_1^{s_4}\ldots\mathbf{f}_1^{s_{n-3}}\mathbf{f}_0^{s_{n-2}}\mathbf{f}_1^{s_{n-1}}(\mathbf{x}_\nu) = \mathbf{f}_1^{s_2}\mathbf{f}_0^{s_3}\mathbf{f}_1^{t_4}\ldots\mathbf{f}_1^{t_{n-3}}\mathbf{f}_0^{s_{n-2}}\mathbf{f}_1^{s_{n-1}}(\mathbf{x}_\nu)$

where s_{2k}, t_{2k} are in the same class A of π, for $k = 1, 2, 3, \ldots$.

Consequently, if $\pi_1 \neq \pi_2$, then $[\Sigma(\pi_1)] \neq [\Sigma(\pi_2)]$. Furthermore, if $\{\pi_i\}_{i\in I}$ is an indexed subfamily of Π_ω, then it is quite direct to verify that $\bigvee_{i\in I}[\Sigma(\pi_i)] = [\Sigma(\bigwedge_{i\in I}\pi_i)]$ and $\bigwedge_{i\in I}[\Sigma(\pi_i)] = [\Sigma(\bigvee_{i\in I}\pi_i)]$.

THEOREM 8: *$\mathscr{L}(\langle 2\rangle)$ contains the dual of Π_ω as a complete sublattice.*

Proof: Let $\mathbf{f}_0$ be the single binary operation symbol, and define $\mathbf{q}_1$ to be the polynomial $\mathbf{f}_0(\mathbf{x}_0, \mathbf{x}_0)$. Then recursively define $\mathbf{q}_{n+1}$ to be the polynomial $\mathbf{f}_0(\mathbf{x}_0, \mathbf{q}_n)$, and put $\mathbf{p}_n = \mathbf{f}_0(\mathbf{q}_n, \mathbf{f}_0(\mathbf{x}_0, \mathbf{x}_0))$.

The components of a polynomial $\mathbf{p}$ (see [2]) in $\mathbf{P}^{(\omega)}(\langle 2\rangle)$ form a connected dyadic tree T_p under the relation 'is a component of'. Let the property $\not{p}$ of polynomials $\mathbf{p}$ in $\mathbf{P}^{(\omega)}(\langle 2\rangle)$ be: T_p has a connected subtree of the form shown in Figure 1.

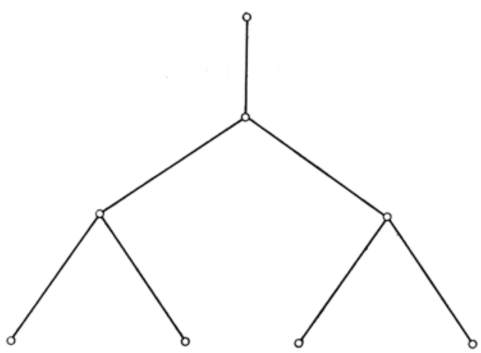

Figure 1.

Note that no $\mathbf{p}_n$ has $\mathscr{p}$, whereas any substitution into $\mathbf{p}_n$ which replaces $\mathbf{x}_0$ by a non-variable yields a polynomial with $\mathscr{p}$, and any substitution of $\mathbf{p}_n$ into a non-variable polynomial results in a polynomial with $\mathscr{p}$.

Let Ψ be the set of identities $\{\mathbf{p}=\mathbf{q}:\mathbf{p}$ and $\mathbf{q}$ have $\mathscr{p}\} \cup \Delta$, where

$$\Delta=\{\mathbf{p}=\mathbf{p}\colon \mathbf{p}\in\mathbf{P}^{(\omega)}(\langle 2\rangle)\}.$$

It is readily seen that Ψ is closed. For any π in Π_ω, define $\Xi(\pi)$ to be the set of identities $\{\mathbf{p}_n(\mathbf{x}_\nu)=\mathbf{p}_m(\mathbf{x}_\nu)\colon n, m\in A$ for some $A\in\pi$, $m\neq n$, and $\nu<\omega\}$; put $\Sigma(\pi)=\Xi(\pi)\cup\Psi$. Since $\Xi(\pi)\cap\Psi=\emptyset$, it follows that $\pi_1\neq\pi_2$ implies $\Sigma(\pi_1)\neq\Sigma(\pi_2)$. Using the property $\mathscr{p}$ and the discussion in the previous paragraph it easily follows hat $\Sigma(\pi)$ is closed; furthermore the $\Sigma(\pi)$ exhaust the subinterval $(\Sigma(\pi_0)\cup\Psi, \Psi)$, where π_0 is the supremum of Π_ω. Hence $(\Sigma(\pi_0)\cup\Psi, \Psi)$ is isomorphic to the dual of Π_ω.

THEOREM 9: *If $\mathfrak{m}_0$ is infinite, then $\mathscr{L}(\tau)$ contains a copy of the dual of $\Pi_{\mathfrak{m}_0}$ as a complete sublattice.*

Proof: One merely partitions the nullary functions and proceeds in an obvious manner.

In summary, let τ be an arbitrary type – if $\mathfrak{m}_1\geqslant 2$, then $\mathscr{L}(\tau)$ contains a copy of $\mathscr{L}(\langle 1, 1\rangle)$ as a complete sublattice, and if $\mathfrak{m}_j\neq 0$ for some $j\geqslant 2$, then $\mathscr{L}(\tau)$ contains a copy of $\mathscr{L}(\langle 2\rangle)$ as a complete sublattice. Hence one has the following result.

COROLLARY: *If $\mathfrak{m}_0\geqslant\aleph_0$ or $\mathfrak{m}_1\geqslant 2$ or $\mathfrak{m}_j\geqslant 1$ for some $j\geqslant 2$, then $\mathscr{L}(\tau)$ contains the dual of Π_ω as a complete sublattice.*

In [3] E. Jacobs and R. Schwabauer proved that the lattice $\mathscr{L}(\langle 1\rangle)$ of equational classes of mono-unary algebras is a distributive lattice, and in [4] J. Jezek showed that $\mathscr{L}(\langle 0\rangle)$, $\mathscr{L}(\langle 0, 0\rangle)$ and $\mathscr{L}(\langle 0, 1\rangle)$ were also distributive – however, the fact that $\mathscr{L}(\tau)$ satisfies any special lattice laws is quite exceptional.

THEOREM 10: *If $\mathfrak{m}_0\geqslant\aleph_0$ or $\mathfrak{m}_1\geqslant 2$ or $\mathfrak{m}_j\geqslant 1$ for some $j\geqslant 2$, then $\mathscr{L}(\tau)$ does not satisfy any special lattice identities.*

Proof: From D. Sachs [8] it is known that Π_ω does not satisfy any special lattice laws.

Acknowledgements

The author is indebted to G. Grätzer for his kind encouragement to extend some initial results on $\mathscr{L}(\langle 1, 1\rangle)$ until a full solution of Problem 33 was obtained. Special thanks are due to D. Higgs for many enjoyable conversations on the topic of the paper.

REFERENCES

[1] G. BIRKHOFF, *Lattice Theory*, AMS Colloq. Publ. *25*, third edition (Providence, Rhode Island 1967).
[2] G. GRÄTZER, *Universal Algebra* (Van Nostrand, Princeton, New Jersey 1968).
[3] E. JACOBS and R. SCHWABAUER, *The lattice of equational classes of algebras with one unary operation*, Amer. Math. Monthly *71*, 151–155 (1964).
[4] J. JEŽEK, *Primitive classes of algebras with unary and nullary operations*, Colloq. Math. *20*, 159–179 (1969).
[5] J. KALICKI, *The number of equationally complete classes of equations*, Nederl. Akad. Wetensch. Proc. Ser. A *58*, 660–662 (1955).
[6] J. KALICKI and D. SCOTT, *Equational completeness of abstract algebras*, Nederl. Akad. Wetensch. Proc. Ser. A *58*, 650–659 (1955).
[7] A. KUROSH, *General Algebra* (Chelsea, New York 1965).
[8] D. SACHS, *Identities in finite partition lattices*, Proc. of the AMS *12*, no. 6, 944–945 (1961).
[9] A. TARSKI, *Equational logic and the equational theory of algebras*, Proceedings of the 1966 Hannover Logic Colloqium (North Holland, Amsterdam 1968).

Added in proof. J. Jezek's paper 'On Atoms in Lattices of Primitive Classes', which recently appeared in Comment. Math. Univ. Carolinae, contains a solution to Problem 33 very similar to the one presented in this paper.

University of Waterloo,
Waterloo, Ontario, Canada