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A simple proof of the hereditary undecidability of the theory of lattice-ordered abelian groups

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In 1967 Gurevich [3] published a proof that the class of divisible Archimedean lattice-ordered abelian groups such that the lattice of carriers is an atomic Boolean algebra has a hereditarily undecidable first-order theory. (He essentially showed the reduct of this class to lattices has a hereditarily undecidable first-order theory: on p. 49 of his paper change $z \neq u+v$ to $z \neq u \vee v$ in the definition of $\bar{P}xy$.

In late 1978 the author introduced a construction which is useful for showing that numerous varieties (or quasi-varieties) have hereditarily undecidable theories (see [1]). We will use this to quickly derive the following weakened version of the above result. (For terminology on lattice-ordered abelian groups see [2]).

THEOREM (Gurevich). *The class of lattice-ordered abelian groups (such that the lattice of carriers is an atomic Boolean algebra) has a hereditarily undecidable theory.*

Proof. Let $(G, +, -, 0, \vee, \wedge)$ be a lattice-ordered abelian group which is linearly ordered, and not simple. Let θ be a congruence on G which is neither the least nor largest congruence on G. Given a field F of subsets of some set I, and a subfield F_0 of F, define

$$H = \{f \in {}^I G : f^{-1}(a) \in F, f^{-1}(a/\theta) \in F_0, \text{ for } a \in G\}.$$

H is a subdirect power of G, and the lattice of carriers of H is readily seen to be isomorphic to F.

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For $g \in G$ let $\bar{g}$ be the constant function of ${}^I G$ with value g. Choose $a, b \in G$ with $0 < a < b$ and $\langle 0, a\rangle \in \theta$, $\langle 0, b\rangle \notin \theta$. Let $B = \{f \in [0, \bar{a}] : f$ has a complement in $[0, \bar{a}]\}$. Then $B = H \cap {}^I\{0, a\}$, so $\langle B, \leqslant\rangle \cong \langle F, \subseteq\rangle$ under the map $f \to f^{-1}(a)$. Next let $\hat{B}_0 = \{f \in [0, \bar{b}] : f$ has a complement in $[0, \bar{b}]\}$. Then $\langle \hat{B}_0, \leq\rangle \cong \langle F_0, \subseteq\rangle$ under the map $f \to f^{-1}(b) = f^{-1}(b/\theta)$. Now let $B_0 = \{\bar{a} \wedge f : f \in \hat{B}_0\}$. Then it is easy to verify that $\langle B, B_0, \leq\rangle \cong \langle F, F_0, \subseteq\rangle$ under the map $f \to f^{-1}(a)$. Thus one can interpret the class of Boolean pairs $\langle F, F_0, \subseteq\rangle$ into the class of such H's using two parameters. By [1] the class of Boolean pairs $\langle F, F_0, \subseteq\rangle$ where F is atomic has a hereditarily undecidable theory, proving the theorem. □

Remark. The above proof made no use of the group operations. We really proved (as essentially Gurevich did) that the reduct of lattice-ordered abelian groups to lattices has a hereditarily undecidable theory.

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