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A correction to "Definable principal congruences in varieties of groups and rings"

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In the paper cited above [1], the proofs of Theorems 1.2 and 1.3 are incorrect. We do not know if the results, as stated, are correct. The error lies in assuming that for all \( \omega(x, y, u, v, \bar{z}) \) in \( \Gamma \) one can claim

\[
F_k \models \omega(x, y, u, v, \bar{z})
\]

as at the top of page 154. However for a restricted class of \( \omega \) in \( \Gamma \) this claim holds, and if one replaces the lemma and theorems of § 1 by the following text then one has a result which is sufficiently strong for the study of groups and rings as in § 2.3. (The main results, those of § 2.3, are correct as presented in [1].)

**Lemma 1.** If \( K \) is closed under ultra products, then given formulas \( \{ \phi_i \}_{i \in I} \) and \( \phi \), we have \( K \models \bigwedge_{i \in J} \phi_i \iff \phi \) iff for some finite \( J \subseteq I \), \( K \models \bigwedge_{i \in J} \phi_i \iff \phi \).

**Proof.** (Standard.)

**Definition 2.** Let \( P \) be the set of polynomials \( p(w, z_0, \ldots, z_n) \), \( n < \omega \). For \( P_0 \subseteq P \), a variety \( V \) has \( P_0 \)-projective principal congruences if, for \( a, b, c, d \in A \in V \), \((a, b) \in \theta_A(c, d)\) holds iff

\[
A \models \exists \bar{z}[a = p(e_1, \bar{z}) \land b = p(e_2, \bar{z})]
\]

for some \( p \in P_0 \), where \( \{e_1, e_2\} = \{c, d\} \).

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Two examples of varieties with $P_0$-projective principal congruences follow:

(1) For rings let $P_0 = \{p_n : n \geq 1\}$ where

$$p_n(w, z_0, \ldots, z_{2n+1}) = \sum_{i=0}^{n-1} z_{2i} \cdot (w - z_{2n}) \cdot z_{2i+1} + z_{2n+1}.$$ 

To see that this works let $a, b, c, d \in R,$ $R$ a ring. Then $(a, b) \in \theta_R(c, d)$ iff $a - b$ is in the ideal generated by $c - d$ iff for some $n < \omega$ and some $a_0, \ldots, a_{2n-1}$

$$a - b = \sum_{i=0}^{n-1} a_{2i}(c - d)a_{2i+1}.$$ 

But then

$$a = p_n(c, a_0, \ldots, a_{2n-1}, d, b)$$

$$b = p_n(d, a_0, \ldots, a_{2n-1}, d, b).$$

(2) For groups of finite exponent $e$ let $P_0 = \{p_n : n \geq 1\}$ where

$$p_n(w, z_0, \ldots, z_{n+1}) = \left[ \prod_{i=0}^{n-1} z_i^{-1} \cdot (w \cdot z_i^{-1}) \cdot z_i \right] \cdot z_{n+1}.$$ 

If $G$ is a group of exponent $e$ and $a, b, c, d \in G$ then $(a, b) \in \theta_G(c, d)$ iff $ab^{-1}$ is a product of conjugates of $cd^{-1}$, hence iff for some $n$ and $a_i \in G$,

$$ab^{-1} = \prod_{i=0}^{n-1} a_i^{-1}(cd^{-1})a_i.$$ 

But then

$$a = p_n(c, a_0, \ldots, a_{n-1}, d, b)$$

$$b = p_n(d, a_0, \ldots, a_{n-1}, d, b).$$

THEOREM 3. Let $V$ be a variety with $P_0$-projective principal congruences, for a given $P_0$. Then $V$ has DPC iff for some finite subset $P_0'$ of $P_0$ there is, for each $p(w, z_0, \ldots, z_n) \in P_0'$, a $q(w, z_0, \ldots, z_k) \in P_0'$ and polynomials $q_i(u, v, z_0, \ldots, z_n), 0 \leq i \leq k$, such that $V$ satisfies, for suitable \{w_1, w_2\} = \{u, v\},

$$p(u, z_0, \ldots, z_n) = q(w_1, q_0(u, v, z_0, \ldots, z_n), \ldots, q_k(u, v, z_0, \ldots, z_n))$$

$$p(v, z_0, \ldots, z_n) = q(w_2, q_0(u, v, z_0, \ldots, z_n), \ldots, q_k(u, v, z_0, \ldots, z_n)).$$
Proof. ($\Rightarrow$) Suppose $V$ has DPC. Then from Lemma 1 there must be a $P'_0 \subseteq P_0$ such that $V$ satisfies

\[ (\ast) \quad \forall p \in P_0 \exists \tilde{z}[x = p(u, \tilde{z}) \land y = p(v, \tilde{z})] \]

\[ \Leftrightarrow \forall q \in P'_0 \exists \tilde{z}[x = q(w_1, \tilde{z}) \land y = q(w_2, \tilde{z})] \]

\[ \{w_1, w_2\} = \{u, v\} \]

Given $p(w, z_0, \ldots, z_n) \in P_0$ let $F$ be the free algebra in $V$ freely generated by $u, v, z_0, \ldots, z_n$. In $F$ let $x = p(u, z_0, \ldots, z_n), \ y = p(v, z_0, \ldots, z_n)$. As

\[ F \models \exists \tilde{z}[x = p(u, \tilde{z}) \land y = p(v, \tilde{z})] \]

it follows by (\ast) that for some $q(w, z_0, \ldots, z_k) \in P'_0$,

\[ F \models \exists \tilde{z}[x = q(w_1, \tilde{z}) \land y = q(w_2, \tilde{z})] \]

with $\{w_1, w_2\} = \{u, v\}$. Thus we can choose polynomials $q_1(u, v, z_0, \ldots, z_n) \in F$ such that

\[ F \models x = q(w_1, q_0(u, v, z_0, \ldots, z_n), \ldots, q_k(u, v, z_0, \ldots, z_n), \ldots, q_k(u, v, z_0, \ldots, z_n)) \]

\[ F \models y = q(w_2, q_0(u, v, z_0, \ldots, z_n), \ldots, q_k(u, v, z_0, \ldots, z_n), \ldots, q_k(u, v, z_0, \ldots, z_n)) \]

Of course if two polynomials are equal in $F$ then the corresponding identity holds in $V$.

($\Leftarrow$) Let $a, b, c, d \in A \in V$ with $(a, b) \in \theta_A(c, d)$. Then, for some $p(x, z_0, \ldots, z_n) \in P_0$.

\[ A \models \exists \tilde{z}[a = p(e_1, \tilde{z}) \land b = p(e_2, \tilde{z})] \]

with $\{e_1, e_2\} = \{c, d\}$. Choose $q, q_0, \ldots, q_k$ as in the statement of the theorem. Then, for suitable $\{\tilde{e}_1, \tilde{e}_2\} = \{c, d\}$,

\[ A \models \exists \tilde{z}[a = q(\tilde{e}_1, q_0(\tilde{e}_1, \tilde{e}_2, \tilde{z}), \ldots, q_k(\tilde{e}_1, \tilde{e}_2, \tilde{z})) \land b = q(\tilde{e}_2, q_0(\tilde{e}_1, \tilde{e}_2, \tilde{z}), \ldots, q_k(\tilde{e}_1, \tilde{e}_2, \tilde{z})] \]
so

\[ A \equiv \exists ar{z}[a = q(\bar{e}_1, \bar{z}) \& b = q(\bar{e}_2, \bar{z})]. \]

Thus the formula \( \phi(x, y, u, v) \) given by

\[ \bigvee_{a \in P_0, \{w_1, w_2\} = \{u, v\}} \exists \bar{z}[x = q(w_1, \bar{z}) \& y = q(w_2, \bar{z})]. \]

defines principal congruences in \( V \).

COROLLARY 4. Let \( V(K) \) be a variety with \( P_0 \)-projective principal congruences. Then \( V(K) \) has DPC iff \( Q(K) \) has DPC.

Proof. The direction \( (\Rightarrow) \) is clear. For \( (\Leftarrow) \) just repeat the first part of the proof of Theorem 3 as \( F \in Q(K) \).

PROBLEM: For arbitrary \( K \) is it true that \( Q(K) \) has DPC implies \( V(K) \) has DPC?

REFERENCES


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