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Closure Homomorphisms

S. BURRIS

University of Waterloo, Waterloo, Ontario, Canada

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The theory of closure homomorphisms will be patterned after the usual theory of homomorphisms for algebraic systems (cf. [3], [4]). In this paper we will see that the closure congruences submit to a simple axiomatization, and also if the closure space is algebraic, then we have a fundamental representation theorem.

A *closure space* (see [2]) is an ordered pair $(\mathbf{C}, S)$ where $\mathbf{C}$ is a closure operator on the set S (i.e., $\mathbf{C}$ is an extensive, monotone, and idempotent set mapping). If θ is a homomorphism from an algebra $(S, \mathscr{F})$ into an algebra $(S_1, \mathscr{F}_1)$ (for the definitions see [3], [4], etc.) and $(\mathbf{C}, S)$ and $(\mathbf{C}_1, S_1)$ are the *canonical* closure spaces associated with $(S, \mathscr{F})$ and $(S_1, \mathscr{F}_1)$ respectively (i.e., $\mathbf{C}(P)$ is the smallest subalgebra containing P, etc.), then we have the identity

$$\theta[\mathbf{C}(P)] = \mathbf{C}_1[\theta(P)] \quad \text{for all} \quad P \quad \text{in} \quad S.$$

We abstract and define a *closure homomorphism* to be a mapping θ from a closure space $(\mathbf{C}, S)$ to $(\mathbf{C}_1, S_1)$ satisfying the identity $\mathbf{C}_1\theta = \theta\mathbf{C}$. [It is interesting to note that had we started out with relational structures $(S, \mathscr{R})$ (see [3]) instead of algebras $(S, \mathscr{F})$, the corresponding condition would have been $\theta\mathbf{C} \subset \mathbf{C}_1\theta$, which for additive closures is precisely the study of continuous functions).]

I. AXIOMATIZATION OF CLOSURE CONGRUENCES

For any mapping θ from S into S_1 we know that $\theta^{-1}\theta$, considered as a subset of $S \times S$, is an equivalence relation. The equivalence relations on a closure space $(\mathbf{C}, S)$ which we form in this manner from closure homomorphisms will be called *closure congruences*, and the family of closure congruences for $(\mathbf{C}, S)$ will be denoted by $\mathfrak{K}(\mathbf{C}, S)$.

In the study of a general algebra $(S, \mathscr{F})$ we find that congruences are intrinsically characterized as equivalence relations which are subalgebras of the product algebra $(S, \mathscr{F}) \times (S, \mathscr{F})$ (see [3]). The following theorem will give an intrinsic characterization of closure congruences. [Where we consider an equivalence relation E to induce a set mapping $E(P) = \{y : (x, y) \in E$ for some x in $P\}$.]

LEMMA 1. *Let E be the equivalence relation associated with the closure homomorphism $\theta : (\mathbf{C}, S) \to (\mathbf{C}_1, S_1)$, (i.e.,$E = \theta^{-1}\theta$). Then $E\mathbf{C}E = E\mathbf{C}$.*

Proof. $E\mathbf{C}E = \theta^{-1}\theta\mathbf{C}\theta^{-1}\theta = \theta^{-1}\mathbf{C}_1\theta\theta^{-1}\theta = \theta^{-1}\mathbf{C}_1\theta = \theta^{-1}\theta\mathbf{C} = E\mathbf{C}$.

By simple examples we can show that not every equivalence relation will satisfy such an identity, so the above lemma gives a genuine restriction on the equivalence relations which are congruences.

LEMMA 2. *If $(\mathbf{C}, S)$ is a closure space and E is an equivalence relation on S, then the following are equivalent:*

(i) $E\mathbf{C}E = E\mathbf{C}$

(ii) $\mathbf{C}E\mathbf{C} = E\mathbf{C}$

(iii) $\mathbf{C}E \subset E\mathbf{C}$

Proof. From $E\mathbf{C}E = E\mathbf{C}$ follows $E\mathbf{C}E\mathbf{C} = E\mathbf{C}\mathbf{C} = E\mathbf{C}$, and since $E\mathbf{C} \subset \mathbf{C}E\mathbf{C} \subset E\mathbf{C}E\mathbf{C}$, we see that (i) $\Rightarrow$ (ii). Clearly (ii) $\Rightarrow$ (iii), and from $\mathbf{C}E \subset E\mathbf{C}$ follows $E\mathbf{C}E \subset EE\mathbf{C} = E\mathbf{C}$, so (iii) $\Rightarrow$ (i).

LEMMA 3. *Let $(\mathbf{C}, S)$ be a closure space and E an equivalence relation on S such that $E\mathbf{C}E = E\mathbf{C}$. Let $S_1 = S/E$ and define $\mathbf{C}_1$ to be $\mu\mathbf{C}\mu^{-1}$, where μ is the canonical map from S onto S_1. Then $\mathbf{C}_1$ is the unique closure on S_1 such that μ is a closure homomorphism with congruence E.*

Proof. $\mathbf{C}_1$ is clearly extensive and isotone. Also $\mathbf{C}_1{}^2 = \mu\mathbf{C}\mu^{-1}\mu\mathbf{C}\mu^{-1} = \mu\mathbf{C}E\mathbf{C}\mu^{-1} = \mu E\mathbf{C}\mu^{-1} = \mu\mu^{-1}\mu\mathbf{C}\mu^{-1} = \mu\mathbf{C}\mu^{-1} = \mathbf{C}_1$, so $\mathbf{C}_1$ is idempotent. To show that μ is a homomorphism we note that $\mu\mathbf{C} = \mu\mu^{-1}\mu\mathbf{C} = \mu E\mathbf{C} = \mu E\mathbf{C}E = \mu\mathbf{C}E = (\mu\mathbf{C}\mu^{-1})\mu = \mathbf{C}_1\mu$. The uniqueness of $\mathbf{C}_1$ is straightforward.

THEOREM 1. *If $(\mathbf{C}, S)$ is a closure space and E is an equivalence relation on S, then $E \in \mathfrak{K}(\mathbf{C}, S)$ if and only if $E\mathbf{C}E = E\mathbf{C}$.*

II. A REPRESENTATION THEOREM

LEMMA 4. *Let $E \in \mathfrak{K}(\mathbf{C}, S)$ and P, Q be subsets of S satisfying $E(P) \subset E(Q)$; then for any $x \in \mathbf{C}(P)$ there is a $y \in \mathbf{C}(Q)$ such that xEy.*

Proof. Note that $\mathbf{C}(P) \subset \mathbf{C}E(P) \subset \mathbf{C}E(Q) \subset E\mathbf{C}(Q)$.

LEMMA 5. *Suppose E is a congruence for the closure space $(\mathbf{C}, S)$ where $\mathbf{C}$ is algebraic* (i.e., $\mathbf{C}(P) = \bigcup\{\mathbf{C}(Q) : Q \subset P, Q \text{ finite}\}$). *Then there is an abstract algebra $(S, \mathscr{F})$ whose canonical closure is $\mathbf{C}$ and such that E is a congruence for $(S, \mathscr{F})$ in the sense of general algebra, and $\mathscr{F}$ consists solely of symmetrical operations. [An operator $f(x_1, ..., x_n)$ is symmetrical if its value is invariant under permutations of the arguments.]*

Proof. First, well-order S in such a manner that if $x < y$ and $x \not{E} y$, then for x_1Ex and y_1Ey we have $x_1 < y_1$. For each finite set Q in S and each $x \in \mathbf{C}(Q)$ define the map $f_{Q,x}$ by:

$$\begin{aligned} f_{Q,x}(x_1, ..., x_n) &= x \quad \text{if} \quad \{x_1, ..., x_n\} = Q \qquad [\text{where } n = \operatorname{Card}(Q)] \\ &= y \quad \text{if} \quad \{x_1, ..., x_n\} \neq Q,\ x_iEq_i, \text{ where} \\ &\qquad\quad \{q_1, ..., q_n\} = Q, \text{ and } y \text{ is an element} \\ &\qquad\quad \text{in } \mathbf{C}(\{x_1, ..., x_n\}) \text{ which is equivalent to } x \\ &= \inf\{x_1, ..., x_n\} \quad \text{otherwise.} \end{aligned}$$

In this definition we consider $\{x_1, ..., x_n\}$ to be a set (rather than an n-tuple). Also there is usually more than one candidate for y—select any one (the existence of such a y is guaranteed by Lemma 4). Clearly each $f_{Q,x}$ is a symmetric operator, and if each argument is replaced by an equivalent element, the values of the operator are equivalent (with respect to E). Then, considering the elements of $\mathbf{C}(\phi)$ as nullary operations, we obtain the desired algebra by letting

$$\mathscr{F} = \{f_{Q,x} : Q \subset S, x \in Q\} \cup \mathbf{C}(\phi).$$

THEOREM 2. *Let θ be a homomorphism from the algebraic closure space $(\mathbf{C}, S)$ onto the closure space $(\mathbf{C}_1, S_1)$. Then we can "fit" the spaces $(\mathbf{C}, S)$ and $(\mathbf{C}_1, S_1)$ with abstract algebras $(S, \mathscr{F})$ and $(S_1, \mathscr{F}_1)$ such that* (i) *θ is a homomorphism from $(S, \mathscr{F})$ to $(S_1, \mathscr{F}_1)$, and* (ii) *the algebraic structures induce the respective closures.*

Proof. Consider the congruence $\theta^{-1}\theta$ on $(\mathbf{C}, S)$ and apply Lemma 5 to obtain an algebra $(S, \mathscr{F})$ which induces $(\mathbf{C}, S)$ and for which $\theta^{-1}\theta$ is a congruence for $(S, \mathscr{F})$. Then define an algebra $(S_1, \mathscr{F}_1)$ by requiring that $g \in \mathscr{F}_1$ iff there is an $f \in \mathscr{F}$ such that

$$\theta f(x_1, ..., x_n) = g[\theta(x_1), ..., \theta(x_n)]$$

for all $x_1, ..., x_n \in S$. With this the theorem follows easily.

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