The first order theory of Boolean algebras with a distinguished group of automorphisms

Stanley Burris

The theory of Boolean algebras with a sequence of distinguished ideals is known to be decidable (Rabin [4], 1969), and the theory of Boolean algebras with a distinguished subalgebra is hereditarily undecidable (Rubin [5], 1976). In this paper we look at, for a group \((G, \cdot, e)\), the class \(\mathcal{B}(G)\) of algebras \(\langle B, \lor, \land, \lnot, 0, 1, (g)_{g \in G} \rangle\) which satisfy, for \(g, h \in G\),

1. \(\langle B, \lor, \land, \lnot, 0, 1 \rangle\) is a Boolean algebra
2. \(g(x \lor y) = g(x) \lor g(y)\)
   \(g(x') = g(x)\)
   \(g(h(x)) = (g \cdot h)(x)\)
   \(e(x) = x\).

We will let \(B_G\) denote such an algebra, and \(B\) the reduct to the Boolean operations. Thus \(B_G\) is a Boolean algebra \(B\) with a group \(G\) of automorphisms acting on \(B\). The only result on the theory of \(\mathcal{B}(G)\) that we know of is due to Wolf ([7], 1975): If \(G\) is a finite solvable group then \(\mathcal{B}(G)\) has a decidable theory. His proof is based on Arens and Kaplansky’s theorem that a finite solvable group of homeomorphisms acting on the Boolean space of a countable Boolean algebra has a fundamental domain. We take another approach, introducing monadic algebras into the study, and arrive at a more general result.

**THEOREM 1.** Let \(G\) be a finite group. Then \(\mathcal{B}(G)\) is a finitely generated discriminator variety and consequently has a decidable first order theory.

**Proof.** Let us define a unary term \(c(x)\) by

\[ c(x) = \bigvee_{g \in G} g(x). \]
As the identity function belongs to \( G \) it is clear that

\[
x \leq c(x),
\]

and since automorphisms are order preserving we have

\[
x \leq y \rightarrow c(x) \leq c(y).
\]  

(C2)

If \( h \) is any member of \( G \) then

\[
h(c(x)) = h\left( \bigvee_{g \in G} g(x) \right)
\]

\[
= \bigvee_{g \in G} h(g(x))
\]

\[
= \bigvee_{g \in G} g(x)
\]

\[
= c(x);
\]

hence

\[
c(c(x)) = c(x)
\]  

(C3)

holds. Next note that

\[
c(x \vee y) = \bigvee_{g \in G} g(x \vee y)
\]

\[
= \bigvee_{g \in G} (g(x) \vee g(y))
\]

\[
= \left( \bigvee_{g \in G} g(x) \right) \vee \left( \bigvee_{g \in G} g(y) \right),
\]

which establishes

\[
c(x \vee y) = c(x) \vee c(y).
\]  

(C4)
Finally

\[ c(c(x')) = c\left( \bigwedge_{g \in G} g(x') \right) \]
\[ = c\left( \bigwedge_{g \in G} g(x') \right) \]
\[ = \bigvee_{h \in G} h\left( \bigwedge_{g \in G} g(x') \right) \]
\[ = \bigvee_{h \in G} \left( \bigwedge_{g \in G} hg(x') \right) \]
\[ = \bigvee_{h \in G} \left( \bigwedge_{g \in G} g(x') \right) \]
\[ = \bigwedge_{g \in G} g(x') \]

shows that we also have

\[ c(c(x')) = c(x'). \quad (C5) \]

For any algebra \( B_G \) in \( \mathfrak{A}(G) \) let \( B_c \) be the corresponding algebra \( (B, \vee, \wedge, ', 0, 1, c) \). Properties (C1)–(C5) say that \( B_c \) is a monadic algebra, and the closed elements of \( B_c \) are precisely the fixed points of \( G \) acting on \( B \).

Let \( + \) be the symmetric difference operation on \( B \), that is

\[ x + y = (x \wedge y') \vee (x' \wedge y). \]

The congruences of \( B_G \) are readily seen to correspond to the ideals \( I \) of \( B \) which are closed under the action of \( G \). For if \( \theta \) is a congruence of \( B_G \) then certainly \( 0/\theta \) is an ideal of \( B \), and if \( (a, 0) \in \theta \) then \( (g(a), g(0)) = (g(a), 0) \in \theta \) for \( g \in G \), so \( 0/\theta \) is closed under \( g \). Conversely given an ideal \( I \) of \( B \) closed under the action of \( G \) the corresponding congruence \( \theta \) of \( B \) defined by \( (a, b) \in \theta \) iff \( a + b \in I \) is also a congruence for \( B_G \) as \( (a, b) \in \theta \) implies \( a + b \in I \), so \( g(a) + g(b) = g(a + b) \in I \), hence \( (g(a), g(b)) \in \theta \) for \( g \in G \).

The congruences of \( B_c \) are known (see [3]) to correspond to the ideals of \( B \) which are closed under the operation \( c \). The conditions \( h(x) \leq c(x) = \bigvee_{g \in G} g(x) \), for \( h \in G \), suffice to show that an ideal \( I \) of \( B \) is closed under \( G \) iff it is closed under \( c \), so \( B_G \) and \( B_c \) have the same congruences.
A monadic algebra is subdirectly irreducible iff the closure operator \( c \) satisfies (see [3]) the implication

\[ x \neq 0 \rightarrow c(x) = 1. \]

Now observe that \( \mathcal{BA}(G) \) is locally finite as \( G \) is finite; for if \( X \subseteq B_G \in \mathcal{BA}(G) \) then the subalgebra of \( B_G \) generated by \( X \) is the subalgebra of \( B \) generated by \( X \cup \{g(x) : g \in G, x \in X\} \). Consequently if \( B_G \in \mathcal{BA}(G) \) is such that \( |B_G| > 2^{|G|} \) let \( A \) be a finite subalgebra of \( B_G \) with \( |A| > 2^{|G|} \). If \( a \) is an atom of \( A \) then \( g(a) \) is also an atom. But then \( c(a) < 1 \) as \( |\{g(a) : g \in G\}| \) is less than the number of atoms of \( A \), so \( B_G \) is not subdirectly irreducible.

Since the subdirectly irreducible members of \( \mathcal{BA}(G) \) have cardinality less than or equal to \( 2^{|G|} \) it follows that \( \mathcal{BA}(G) \) is finitely generated, i.e. generated by finitely many finite algebras. Now, as in the case of monadic algebras (see [6]), the ternary term

\[ t(x, y, z) = [x \land c(x + y)] \lor [z \land (c(x + y))^\prime] \]
defines the ternary discriminator on each of the subdirectly irreducible members of \( \mathcal{BA}(G) \), that is for \( a, b, c \) in \( B_G \) we have \( t(a, b, c) = a \) if \( a \neq b, c = c \) if \( a = b \). Thus \( \mathcal{BA}(G) \) is indeed a discriminator variety, and hence has a decidable first order theory (see [2]). □

Remark. If one replaces the finite group \( G \) by a finite monoid \( M \) then \( c(x) \), defined as before, still satisfies (C1)–(C4), but one can readily find a four-element member \( B_M \) of \( \mathcal{BA}(\bar{M}) \), where \( \bar{M} \) is a two-element monoid, such that (C5) fails in \( B_M \); furthermore \( B_G \) is subdirectly irreducible but not simple, so \( \mathcal{BA}(\bar{M}) \) has an hereditarily undecidable theory (by Theorem 9.1 of [1]). However for any finite monoid \( M \) the subvariety of \( \mathcal{BA}(M) \) defined by (C5) is a finitely generated discriminator variety; hence it has a decidable theory. It would be interesting to know for which finite monoids the theory of \( \mathcal{BA}(M) \) is decidable.

Now we will apply our first theorem to construct what we think is the first example of a decidable congruence distributive variety which is not a discriminator variety. From [1] we know that such a variety cannot be finitely generated.

**Theorem 2.** The variety \( \mathcal{BA}(\Sigma_\omega Z_2) \) is arithmetical (i.e. congruence distributive and congruence permutable), has a decidable first order theory, but is not semi-simple (i.e. not every subdirectly irreducible member is simple).

**Proof.** Certainly this variety is arithmetical as Boolean algebras are a reduct of
it. Now we claim that for any \( n < \omega \), the reduct of \( \mathcal{A}(\sum_n \mathbb{Z}_2) \) to the language 
\( L_n = \{ \lor, \land, ', 0, 1, \{ g \}_{g \in \sum_n \mathbb{Z}_2} \} \), where we make the obvious identification of \( \sum_n \mathbb{Z}_2 \) with the subgroup of \( \sum_n \mathbb{Z}_2 \) consisting of those elements which are zero on all \( i \geq n \), is just \( \mathcal{A}(\sum_n \mathbb{Z}_2) \). The containment from left to right is clear. For the converse choose a member of \( \mathcal{A}(\sum_n \mathbb{Z}_2) \), and extend it to a member of \( \mathcal{A}(\sum_n \mathbb{Z}_2) \) by letting all the additional automorphisms \( g \), with \( g(i) = 0 \) for \( i < n \), be the identity map. As each \( \mathcal{A}(\sum_n \mathbb{Z}_2) \), \( n < \omega \), has a decidable first order theory by Theorem 1 it follows that the theory of \( \mathcal{A}(\sum_n \mathbb{Z}_2) \) is also decidable (as every reduct to a finite language is decidable).

To finish the proof let \( G = \sum_n \mathbb{Z}_2 \), and let \( B \) be the power set Boolean algebra \( P(G) \). Define the action of \( G \) on \( B \) to be

\[
g(X) = \{ g \cdot h : h \in X \}
\]

for \( g \in G \). Then the smallest non-zero ideal of \( B \) closed under \( G \) is the ideal of all finite subsets of \( G \). Thus \( B_G \) is subdirectly irreducible but not simple, and clearly \( B_G \) is in \( \mathcal{A}(G) \). \( \Box \)

The proof of Theorem 2 obviously goes through for any countable recursive abelian group of finite exponent. Thus the next result stands in sharp relief.

**THEOREM 3.** If \( G \) is not a locally finite group then the first order theory of \( \mathcal{A}(G) \) is hereditarily undecidable.

**Proof.** Let \( B \) be the power set algebra \( P(G) \), and let \( G \) act on \( P(G) \) as in the proof of the previous theorem, giving \( B_G \). Choose finitely many elements \( g_1, \ldots, g_n \in G \) such that \( g_1, \ldots, g_n \) generate an infinite subgroup \( H \) of \( G \). Let \( a = \{ g_1 \} \), and let \( b = H \). Then \( 0 < a < b \), \( a \) is an atom of \( B \), and \( b \) is an infinite subset of \( G \).

Let \( \theta \) be the congruence of \( B_G \) determined by the ideal of finite subsets of \( G \) (this ideal is indeed closed under the action of \( G \)). Given an arbitrary field \( F \) of subsets of a set \( I \) and a subfield \( F_0 \) of \( F \) define the member \( B_G[F, F_0, \theta]^* \) of \( \mathcal{A}(G) \) by

\[
B_G[F, F_0, \theta]^* = \{ f \in F(B_G) : f^{-1}(e) \in F, f^{-1}(e/\theta) \in F_0, \text{ for } e \in B, \text{ and } |f(I)| < \omega \}.
\]

(This is, indeed, a subdirect power of \( B_G \).) For \( e \in B_G \) let \( \bar{e} \) denote the constant function in \( F(B_G) \) with value \( e \). Since \( a \) is an atom of \( B_G \) the interval \( \hat{F} = [0, \bar{a}] \) in \( B_G[F, F_0, \theta]^* \) is order-isomorphic to \( F \) under the map \( f \mapsto f^{-1}(a) \); and as the only elements in the interval \([0, b]\) which are fixed by all of \( g_1, \ldots, g_n \) are the elements...
0 and b, it follows that $F_0$ is order-isomorphic to $\tilde{F}_0 = \{ x \in [0, \tilde{b}] : g_i(x) = x, 1 \leq i \leq n \}$ under the map $f \mapsto f^{-1}(b) = f^{-1}(b/\theta)$, and hence to $\tilde{F}_0 = \{ \tilde{a} \land x : x \in [0, \tilde{b}], g_i(x) = x, 1 \leq i \leq n \}$. From this one can check that $\langle F, F_0, \subseteq \rangle \cong \langle \tilde{F}, \tilde{F}_0, \subseteq \rangle$. Thus one can easily write down a first order interpretation of the Boolean pair $(F, F_0, \subseteq)$ into $G_\theta[F, F_0, \theta]^*$, and as the theory of such pairs is hereditarily undecidable (see [1]), so is the theory of $BA(G)$. □

An interesting problem is to determine for precisely which groups G the theory of $BA(G)$ is decidable. Of course the groups must be recursive and locally finite; if finitely presented it must have a solvable word problem. So far we do not even know if the theory of $BA(Z_\infty)$ is decidable.

REFERENCES


University of Waterloo
Waterloo, Ontario
Canada