Contributions of

The Logicians.

Part I.
From Richard Whately to William Stanley Jevons

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Preface

As an undergraduate I read portions of Boole's 1854 classic, The Laws of Thought, and came to the conclusion that Boole was rather inept at doing Boolean algebra. Much later, when studying the history of logic in the 19th century, a conversation with Alasdair Urquhart led me to the 1976 book of Theodore Hailperin that shows why Boole's methods work. This was when I finally realized that trying to superimpose Boolean algebra, or Boolean rings, on Boole's work was a real mistake. Boole uses the ordinary number system, and not a two-element algebra.

With the exception of Hailperin's book I do not know of a single history of logic or mathematics that properly explains Boole's work. Furthermore, even though the books and papers devoted to the development of mathematical logic in the mid 1800s are readily available today, the essential facts have not been written up in a concise yet comprehensive form for a modern audience. These notes intend to round out the picture with a brief but substantial account of the transition from Aristotelian logic to mathematical logic, starting with Whately's revival of Aristotelian logic *The Elements of Logic* in 1826, continuing through the work of De Morgan, Boole and Jevons, and concluding with modern versions of equational proof systems for Boolean algebra, Boolean rings, and a discussion of Hailperin's exposé of the work of Boole.

Many of the sections are devoted to discussing a single book. Within such sections plain page number references (without a source cited) apply to the book being discussed in the section.

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Prelude: The Creation of an Algebra of Logic

The successful introduction of mathematical techniques to handle traditional Aristotelian logic was based entirely on **the development of an equational logic for classes**. We will trace this development through the works of De Morgan, Boole and Jevons.[1] By studying equations involving the operations of union, intersection, and complement, they created an algebraic approach that led to an extension of traditional Aristotelian logic. The development of logic *for* mathematics was initiated by efforts to find the axioms and rules of inference needed for an equational logic of classes.

Some of the most fundamental perspectives of modern logic were not available to these pioneers. Above all they did not have a clear separation of logic into its syntatic and semantic components. For example, in 1874 Jevons says that A = B and B = A are really just "one equation accidentally written in two different manners." The modern viewpoint says that the two equations are different (as strings of symbols), but they have the same truth value under any given interpretation.

De Morgan was fascinated by the study of equality insofar as it led to new forms of categorical propositions, but he did very little in the way of an equational logic formulation. The famous De Morgan laws he stated as ([**4**], p. 118):

The contrary of PQ is p,q; that of P,Q is pq.

Although he had many ideas regarding how one could make use of symbolism to give a fresh approach to syllogistic logic, it is curious that he did not use a symbol for equality in his work and continue to develop an equational logic of classes. Instead this was done, in a rather baffling manner, by Boole. A transparent approach would have to wait till the early 1860s, with the work of De Morgan's student Jevons.

De Morgan was an inspiration for his younger friend, the school teacher Boole, as well as for Jevons. De Morgan's feud with the Scottish philosopher Sir William Hamilton of Edinburgh[2] caused Boole to put aside his work in traditional mathematics (differential equations, the calculus of variations, etc.) in 1847 in order to apply algebra to logic. Revising this initial work on logic would be Boole's main interest till 1854. Then he returned to traditional mathematics.

Boole views classical logic as reasoning about classes: given a collection $\mathcal{P}_1, \ldots ,\mathcal{P}_n$ of propositions about classes one wants to find propositions $\mathcal{P}$ about (some of) these classes that follow from $\mathcal{P}_1, \ldots ,\mathcal{P}_n$. Boole's new approach to logic is built on two beliefs:

1. the meaning of such propositions can be expressed by equations, and
2. one can apply the symbolic methods of an augmented ordinary algebra to the equations of the premises to find the equations of the conclusions.

The first item is handled convincingly by Boole, at least for universal propositions, but the second item is not. In the 1847 book he simply describes his algebraic methods and employs them. Seven years later, in his 1854 book, he is caught up in trying to explain why it is permissible to use terms in his equations that have no interpretation as classes. His explanations were, and still

[1]The names Boole and De Morgan are well known in logic circles, but not that of Jevons. Jevons is much better remembered for his contributions to economics—he is considered one of the fathers of mathematical economics, introducing the methods of calculus to the subject. In this arena one of his least successful and most memorable contributions was to tie economic cycles to sunspot cycles.

[2]Not to be confused with the mathematician of quaternion fame, Sir William Hamilton of Dublin.

are, most unconvincing. In essence his defence was just that he was using the 'symbolic method', and the requests for clarification were mainly met with a number of new examples showing how to use his system. Boole discovered that ordinary algebra leads to a powerful algebraic system for analyzing logical arguments, but he really did not know why it worked. Indeed, it seems that the fact that it did work in many examples was quite enough to convince Boole that it would work in general—a masterful application of inductive reasoning to provide an equational approach to classical deductive reasoning. The fact that his methods provide an equational logic that can indeed correctly handle arguments about classes was not established until Theodore Hailperin's book appeared in 1976.[3] Hailperin uses **signed multisets** to extend the collection of classes, after identifying classes with idempotent multisets. In this setting Boole's operations become meaningful, namely $class + class$ may not be a class, but it is always a signed multiset.

The justification of Boole's system likely requires more sophistication in modern algebra than was available in the 19th century. This would lead Jevons in the 1860s to create an alternative algebra of logic, essentially what is now called Boolean algebra.[4] Boole's work is a clever interplay between traditional logic and the algebra of numbers. With Jevons, logic abandons the ties to numbers and joins its new companion, 'Boolean' algebra, for a mature and enduring relationship.

To have a good idea of where Boole and De Morgan were starting from we begin these notes with an overview of the influential 1826 logic text of Richard Whately. The notion of a **class** was well established in the traditional literature on Aristotelian logic, as well as the use of a single letter to denote a class, for example, using M to denote the class of men. However, using symbols to describe combinations of classes, for example, using $\mathrm{A} + \mathrm{B}$ for the union of A and B, was not part of the traditional literature of either logic or mathematics.

Variables and Constants

In **modern equational logic**[5] we use two kinds of symbols as simple names for the objects in the universe of discussion, namely **variables**, usually letters $x, y, z, \ldots$ from the end of the alphabet, and **constants**, usually letters $a, b, c, \ldots$ from the beginning of the alphabet, as well as the numerals $0, 1, \ldots$. The reason for the two kinds of symbols is that we treat the variables in an equation as *universally* quantified, but the constants are not quantified. For example the equation

$$xa = ax$$

says that everything (in the universe of discussion) commutes with the object designated by a, whereas the equation

$$xy = yx$$

says that any two objects commute.

This partition of the simple name symbols into two kinds does not apply to the work on logic from the last century. Except for the constants 0 and 1 they treat the simple name symbols as being capable of being universally quantified in some equations and not quantified in others. To distinguish which is the case they use the word **Law** to designate an equation in which the simple

[3] An expanded 2nd edition appeared in 1986.

[4] De Morgan was quick to praise Boole's work, but he apparently put little serious effort into trying to understand it. Jevons, however, was determined to make sense of Boole's algebraic approach, and he modified it in such a way that it could be readily justified. It is essentially the interpretations and equational axioms of Jevons that most of my colleagues think is the contribution of Boole, the modern Boolean algebra. But it most certainly is not what Boole did, nor does it seem that Boole had much sympathy for this 'Boolean' algebra.

[5] Jevons 1880 book [**10**] has a chapter titled Elements of Equational Logic, meaning his reworking of Boole's algebra of logic.

name symbols are to be treated as universally quantified. For example, one can have the law

$$xy = yx$$

which says that for any two classes the intersection does not depend on the order in which one takes the classes. On the other hand one can have a syllogistic argument

$$\begin{array}{c} \text{All } x \text{ is } y \\ \text{All } y \text{ is } z \\ \hline \text{All } x \text{ is } z \end{array}$$

where it is understood that the x in the first premiss and the conclusion refer to the same class, etc. When this is translated into equations in the algebra of logic, say by

$$\begin{array}{rcl} xy & = & x \\ yz & = & y \\ \hline xz & = & x, \end{array}$$

again the x in the first equation and the last equation refer to the same class, etc. Thus for each of these equations the simple name symbols are not regarded as universally quantified.

The words *variable* and *constant* were not used by De Morgan, Boole or Jevons, so to prevent any modern nuances from slipping into the reading we will also avoid these words when commenting on quoted portions of their texts. Instead, if we do not use the same nomenclature as the quoted text then we will simply use the word SYMBOL, in the indicated font.

Semantics of Class Symbols and the Existential Import of Universal Statements

We would like to give an evaluation of the ability of Boole's algebra of logic to capture correct reasoning about classes. For this we need to know how Boole interprets SYMBOLS, and unfortunately this is not completely clear. However we can make a very educated guess as to what he had in mind.

Both Boole and De Morgan introduce the universe among the classes they consider. In 1847 they both have some notion of an empty class, but it does not quite have the clarity of our modern concept. They mix discussion about X applying to no things, with X being nonexistent, or nothing. In 1854 Boole gives the modern definition of an empty class as a class with no things in it, but he is not willing to use it as freely as we do, on an equal footing with other classes. This clash leads to some bookkeeping to sort out the differences between the modern logic of classes and Boole's logic of classes.

In **modern semantics** we can interpret a SYMBOL as any subclass of the universe, including the empty class. But this semantics is in apparent conflict with the traditional **conversion per accidens**

$$\begin{array}{rl} & \text{All X is Y} \\ \therefore & \text{Some Y is X} \end{array}$$

which clearly gives existential import to the universal premiss. Furthermore both De Morgan and Boole apparently accept the related conversion

$$\begin{array}{rl} & \text{All not-X is Y} \\ \therefore & \text{Some Y is not-X.} \end{array}$$

How can X be empty, or the universe, if we accept these? One approach to resolve this difficulty is to declare that 'All X is Y' is false if X is empty. This convention certainly runs against our modern training, but it was evidently the norm during the medieval period (see Kneale and Kneale [**13**], page 264). De Morgan says ([**4**], page 120) such an assertion does not have a clear meaning when X is empty, and he declines to assign either true or false to it. He says that in case both X and Y are empty that he will

> . . . not attempt to settle what nonexisting things agree or disagree.

In 1847 Boole translates 'All X is Y' as the equation $xy = x$, and when X is empty this becomes the equation '$0y = 0$', which is true in his system. However Boole does not explicitly discuss the meaning of 'All X is Y' when the subject or predicate are empty, and consequently he does not discuss the truth or falsity of such a statement.[6] The closest Boole comes to discussing the role of the empty class in his semantics is on page 65 of [**1**]:

> It may happen that the simultaneous satisfaction of equations thus deduced, may require that one or more of the elective symbols should vanish. This would only imply the nonexistence of a class: it may even happen that it may lead to a final result of the form
>
> $$1 = 0,$$
>
> which would indicate the nonexistence of the logical Universe. Such cases will only arise when we attempt to unite contradictory Propositions in a single equation.

Thus a conclusion $1 = 0$ would mean that the premises were contradictory.

The second obvious way to resolve the problem with conversion per accidens is to use what we call **restricted semantics**, namely one is allowed to interpret a SYMBOL as any subclass of the universe except the empty class and the universe. Kneale and Kneale ([**13**], page 408) note that Aristotle did not permit terms to denote either the empty class or the universe, but it is doubtful that Aristotle would have been so generous as to accept all the interpretations allowed by our restricted semantics. For example, traditional Aristotelian logic does not allow the contrary of a term, e.g. 'not-man', to be the subject of a universal statement.

Although we are not clear as to the precise semantics of Boole or De Morgan, the restricted semantics seems to fit remarkably well. After declining to speculate as to whether 'All X is Y' is true or false when X does not exist, De Morgan makes the condition of existence (i.e., nonemptiness) of the terms a precondition for using a categorical proposition in a syllogism.[7]

De Morgan had stated ([**4**], page 55) that no term of a proposition was to name the universe. This was not because it created problems with the meaning of propositions, but rather because it led to trivial simplifications of the premises that would eliminate the reference to such terms. However once he excludes the empty class from being an interpretation of the SYMBOLS, it seems clear that he must also exclude the universe—to be able to use contraries of terms on an equal footing with terms.

Putting the above pieces together it seems that the restricted semantics is in excellent harmony with the writings of Boole and De Morgan. All their simple inferences, including the conversions, are valid under the restricted semantics. There is one problem with Boole's classification of the valid syllogisms, given that he, like De Morgan, accepts contrary terms such as not-X on an equal footing with the term X. From the 2nd Figure AA premises we have, under the restricted semantics,

[6]The clear decision to admit the empty class as an acceptable interpretation of the subject of a categorical proposition seems to have originated with C.S. Peirce in his 1880 paper [**16**] on the algebra of logic. Peirce looks at the examples '[All/Some] lines [are/are not] vertical' and pictures four different scenarios, two of which have no vertical lines. He assigns, in modern fashion, truth values to each of the four Aristotelian categorical statements in each of these four cases. Schröder adopts Peirce's conventions in the second volume (1891) of his influential Algebra der Logik, and goes on to clarify the situation when the predicate does not apply to anything, saying that 'All X is Y' will mean the same as $X \subseteq Y$, which is precisely the modern interpretation. Schröder says that although dealing with assertions about an empty subject is not an issue in everyday life, in the scientific community one has to constantly deal with the possibility that there are no entities fitting a given description.

[7]This leads to some discussion about whether or not the conclusion of a syllogism should include the phrase 'provided the middle term exists'.

the valid syllogism

All Z is Y
All X is Y
∴ Some not-X is not-Z.

However in 1847 Boole says ([**1**], page 35) that there is no conclusion to be drawn from these premises. De Morgan discusses precisely this syllogism in some detail[8] in his 1847 book [**4**], page 158, as an example of a valid syllogism. We consider this omission an oversight on Boole's part.

The real defect of the restricted semantics is that Boole's main theorem, the Elimination Theorem, is far less powerful than under the modern semantics. There is no simple way to modify the Elimination Theorem to give it the force under restricted semantics that it has under modern semantics.

The Equational Treatment of Particular Statements

Boole originally translated 'Some X is Y' into the equation $v = xy$, and later as $vx = vy$. The treatment of particular propositions as equations provoked strong criticism, starting with Peirce in 1880 and continued by Schröder in 1891. Subsequently an equational approach to particular propositions was considered flawed (see, e.g., Hailperin [**6**], pages 97–98). However, with a small adjustment, it turns out that Boole's method works perfectly well in the context of modern semantics, and almost as smoothly with restricted semantics. (See Appendix 4.)

Terms

Both De Morgan and Boole invent symbolic notation to give names to certain combinations of classes. For example, if A and B denote two classes then they both use the name AB to denote the class of elements common to the two classes. De Morgan calls such names **compound names**, and Boole calls them **functions**. The modern terminology for such names, including the simple name symbols, is **terms**. For De Morgan, compound names for classes are a sidelight to his investigations of the syllogism. He is much more interested in names for combinations of propositions, as well as of binary relations. But for Boole, his functions are central to his algebra of logic.

In modern logic a term that does not include any variables is called a **ground term**, and an equation that does not include any variables is called a **ground equation**. Using this terminology, the translations by Boole and Jevons of an argument into equations yield arguments about ground equations.

Boole, motivated by ordinary algebra, introduced **simple name symbols for terms** such as φ, f, t, V, etc. These would be useful in formulating his general Expansion,[9] Reduction and Elimination Theorems. For example his Expansion Theorem in one SYMBOL is $f(x) = f(1)x + f(0)(1-x)$. In our commentary we like to use $p, q, r, \ldots$ as simple names for terms. It seems a bit curious that De Morgan and Jevons did not introduce simple names for terms.

De Morgan and Jevons both work with the operations of union, intersection and complement in the modern sense. Boole also works with intersection and complement in the modern sense, but he introduces the operations $+$ and $-$ as partial operations on classes. If x denotes the class X and y the class Y then $x + y$ in Boole's system represents the union of X and Y provided X and Y are *disjoint* classes; and $x - y$ represents the difference of X and Y provided Y is *contained in* X.

[8] It is described in his notation as $\mathrm{A}_{\mathsf{I}}\mathrm{A}^{\mathsf{I}}\mathrm{I}^{\mathsf{I}}$. See also pages 105 and 116 of [**4**].

[9] Boole uses the word 'Development' instead of 'Expansion'.

Kneale and Kneale ([**13**], p. 410) claim that Boole does not even allow one to write down $x + y$ unless x and y represent disjoint classes. But this claim is simply incorrect. Although many terms such as $x + y$ may be *uninterpretable*, that does not mean they cannot be written down and used in Boole's system. In his 1854 book Boole strenuously argues for the admittance of uninterpretable terms in the intermediate steps of a symbolic argument—he says that to restrict all steps of equational inference to interpretable terms would destroy much of the value of his system. One sees uninterpretable terms used freely in his Chapter VIII On the Reduction of Systems of Propositions, especially in his favorite procedure to reduce a system of equations $V_1 = 0$, $V_2 = 0$, $\ldots$ to a single equation $V_1^2 + V_2^2 + \cdots = 0$.

The Rule of 0 and 1

Boole took as the foundation of his 1854 treatment of logic a principle that we will call the **Rule of 0 and 1**, namely that a symbolic assertion is correct in the algebra of logic if and only if it is correct in ordinary algebra when the SYMBOLS are restricted to 0 and 1. This remarkable principle is the essence of the modern method of truth tables, except that *Boole does not have a closed two-valued system*—he uses **ordinary arithmetic**.

Let us apply his Rule of 0 and 1 to justify his law $x + y = y + x$, using a modern truth table style presentation:

x	y	$\mathbf{x+y}$	$\mathbf{y+x}$
1	1	**2**	**2**
1	0	**1**	**1**
0	1	**1**	**1**
0	0	**0**	**0**

Fig. **1** Justifying the Commutative Law

Unfortunately Boole did not give such simple detailed applications of this rule. He does state that it justifies his Expansion Theorem. In the one SYMBOL case this is, as mentioned earlier,

$$f(x) \;=\; f(1)x \;+\; f(0)(1-x)\;.$$

A detailed truth table style presentation to apply the Rule of 0 and 1 would look like

x	$\mathbf{f(x)}$	$f(1)x$	$f(0)(1-x)$	$\mathbf{f(1)x+f(0)(1-x)}$
1	$\mathbf{f(1)}$	$f(1)$	0	$\mathbf{f(1)}$
0	$\mathbf{f(0)}$	0	$f(0)$	$\mathbf{f(0)}$

Fig. **2** Justifying the Expansion Theorem

The Rule of 0 and 1 can also be used to check the correctness of arguments. However Boole only did this in one simple case, namely if a is a numerical coefficient that is not 0 then from $at = 0$, where t is a special kind of term called a *constituent*, one can conclude $t = 0$. Had Boole put more emphasis on applications of this Rule of 0 and 1 perhaps truth tables would have been established much earlier in the logic literature.

Semantics of Laws

The semantics of laws must be handled with care in the work of Boole. When he states the laws

$$\begin{aligned} xy &= yx \\ x^2 &= x \end{aligned}$$

one would like to think that they stand on an equal footing. When one has a law like

$$xy = yx$$

this normally means that one can *substitute* terms s, t for the SYMBOLS x, y and conclude that

$$st = ts$$

holds. After all, if this law means that any two objects commute then surely any two objects named by s and t commute.

However Boole's system is deviant in this regard. He has the law $x^2 = x$, but you cannot derive $(x + x)^2 = x + x$. The reason is that Boole's idempotent law applies only to classes, and unfortunately a term like $x + x$ need not refer to a class. On the other hand Boole's rule of 0 and 1 shows that for his law $xy = yx$ one can indeed substitute any terms s, t for x, y.

Substitution

The use of the word *substitution* in the 1800s was different from the modern usage in equational logic. In 1869 Jevons regards his rule of substitution as the sole central principle of reasoning, replacing the traditional dictum de omni et nullo of Aristotle. His usage of substitution is that if A = B then we can replace B by A in any assertion about B and obtain an equivalent assertion. This mainly corresponds with what is called *replacement* in modern equational logic, namely if A = B then A + C = B + C, C + A = C + B, etc. One can view replacement as saying that doing the same thing to both sides of an equation, for example, adding C to the right side of each term, gives an equation that follows from the original equation. This was essentially the sole rule of inference proposed by Boole in 1847, namely

> equivalent operations performed upon equivalent subjects provide equivalent results.

Boole did not give a special name to this rule—he merely said that it was the only *axiom* that was needed. Thus, as far as equations go, we claim that the *the main rule of inference proposed by Boole*[10] *in 1847, and by Jevons*[11] *in 1869, was the modern rule of replacement.*

In modern equational logic we use the word substitution to mean the *uniform* substitution of terms for variables, for example, if we substitute $x+y$ for x in $x^2 = x$ we obtain $(x+y)^2 = x+y$. In modern propositional logic we also use the word substitution in such a uniform sense when we speak of a substitution instance of the tautology $P \to (Q \to P)$. Eighteenth century writers apparently had no word for our modern substitution.

We want to look at Jevons' use of substitution, and also to discuss modern substitution. To sort out this substitution tangle we will adopt the following convention: the word *substitution* will at times be prefixed with a bracketed word to clarify the version of substitution that we are talking about, namely **[Jevons] substitution** or **[modern] substitution**. This will allow us to make

[10] In his 1854 book Boole formulated this as: adding or subtracting equals from equals gives equals. But in 1854 this was no longer the guiding principle of inference for Boole—instead it was the Rule of 0 and 1.

[11] Jevons' substitution is stronger than replacement, for example, one can derive the transitive rule for equality from this. Jevons' substitution for equations, plus the reflexive law for equality, is equivalent to replacement plus the reflexive, symmetric and transitive laws for equality.

exact quotes from Jevons' work, treating the bracketed items as editorial comments, and at the same time allow us to use the word substitution in commentary with the modern sense.

Modern Equational Proof Systems

In Appendix 1 we give a a modern equational proof system $\mathcal{BR}$ for **Boolean rings**, and in Appendix 2 such a proof system $\mathcal{BA}$ for **Boolean algebra**. Boole's work is much closer to the study of ground equations for Boolean rings than to the study of such equations for Boolean algebra. Jevons work is essentially the study of ground equations for Boolean algebra.

In Appendix 3 we set up a modern proof system $\mathcal{AB}$, based on the work of Hailperin, that is faithful to the methods and interpretation used by Boole insofar as one avoids Boole's treatment of particular statements and his use of division, and one uses modern semantics for the SYMBOLS. One key step is to declare that the SYMBOLS used by Boole for classes are to be regarded as *constant symbols* except in the case of the usual laws of algebra where they refer to ground terms. Thus the law $x^2 = x$ becomes the collection of assertions $a^2 = a$ where a is a constant symbol. But the commutative law $xy = yx$ becomes the collection of assertions $st = ts$ where s and t are any ground terms. Indeed our formalization of Boole's algebra of logic is an equational logic with constant symbols and no variable symbols. Then replacement does become the only rule of inference that one needs to add to the obvious reflexive, symmetric and transitive properties of equality. (These three obvious properties of equality were not mentioned by Boole, but were eventually all recognized by Jevons.)

The use of constant symbols instead of variables in equational arguments is, as noted earlier, faithful to the study of arguments in logic. So the use of constant symbols, but no variable symbols, seems preferable, to emphasize that the rule of [modern] substitution does not apply.

CHAPTER 1

Richard Whately, Archbishop of Dublin (1787–1863)

By the beginning of the nineteenth century the importance of traditional *deductive* logic had already gone through a major decline lasting nearly two centuries, the victim of exaggerated claims. It had been promoted as the ultimate and unique means of finding truth and accumulating knowledge, the perfect tool for scientific discovery. The reaction to these claims had been a major switch to *inductive* logic, championed by Bacon, Descarte, Locke, and Playfair. The traditional logic was viewed as rigid and sterile, whereas the inductive method, of accumulating facts to discover general principles, was considered the hallmark of science. In England, only Oxford University had kept classical logic as part of its university exams. And in the early part of the nineteenth century even Oxford was considering dropping the subject.

The person who perhaps did the most to reverse this trend was Richard Whately of Oxford. He grew up in a family of nine children in London, his father being a church minister. In 1805 he entered Oxford University, took his B.A. from Oxford in 1808 in classics and mathematics, and in 1810 his M.A. In 1811 he was made a fellow of Oriel College, Oxford, and remained at Oxford till 1831.[1] In 1826 he published **Elements of Logic**, a presentation of classical logic essentially based on the *Artis Logicae Compendium* (1691) of Henry Aldridge.

There was really nothing new as far as the development of logic goes in Whately's book, but what was given was presented in a simple and clear manner. His approach to logic was strictly traditional. He praised Aristotelian logic for having realized that the syllogism was the ultimate form of argument, and that all correct syllogistic reasoning could be reduced to the principle of *omni et nullo*, i.e., that what was true in the general situation held in the particular, and what was false in the general situation was false in the particular. He tried to reconcile the conflict with the supporters of inductive logic by pointing out that inductive logic really consisted of two parts, the first concerned with collecting information to determine plausible premises, and then the use of deductive logic to find further information based on those premises. Thus he saw harmony, not conflict, between inductive logic and deductive logic.

In 1831 he became Archbishop of Dublin and left Oxford for good. His **Elements** became the standard logic textbook at Oxford, and was popular throughout England for the rest of the nineteenth century. By 1840 Whately could say that his book had been adopted by all the colleges in America. It is still easily available today.

This excellent presentation of logic was an important anchor point for many future logicians. In particular Augustus De Morgan uses it as a standard reference, George Boole uses Whately's book as his main source when writing his first book on logic, and Charles S. Peirce said that Whately's book had been his introduction to the subject of logic. Perhaps it was the elegant simplicity and clarity of purpose of Whately's work that invited others to try to improve on it.

1. Elements of Logic (1826)

The book opens in a rather unusual manner, by lamenting the poor state of the logic exams at Oxford. They were required of all degree candidates, and the standards had dropped to the point

[1]Except for the four year period 1821–1825 when he married and was a rector of a parish. In 1825 he became a Doctor of Divinity.

of being meaningless. Whately commends the faculty for having rejected the proposal to remove logic altogether, but now he urges them to consider requiring the exam only of those students who wished to graduate with distinction. This way the standards would be upgraded to this elite group of students, and they would receive a solid and meaningful course. In this effort at changing the curriculum he failed.

But the book succeeded. Perhaps the most important goal of Whately was to rescue deductive logic from the excessive burden it carried because of the extravagant promises that had been made regarding its purpose. In the introduction he emphasized that its *only goal* was to deal with the correct forms of argument, not with ascertaining the validity of the premises. And in this respect he deemed it to be a science as much as an art, and worthy of the respect and study given to any science. Also he brought out a new fact, the intimate **connection of logic and language**, how it would be impossible to reason about a subject without language.[2]

Although the study of logic might not improve one's reasoning skills, still it would provide a tool to defend oneself against fallacious arguments of others. He particularly recommended this to his fellow Christians since, as he said, it was clear that the shrewd opposition was indeed making itself well acquainted with logic.

After clarifying the goal of logic in the introduction Whately has an interesting strategy for presenting the subject.

> (1) First he gives a sketchy but engaging *vocabulary building overview* in the chapter Analytical Outline by working from the general nature of logical argument via examples down to the finer structure of symbolic logic. En route he takes the opportunity to introduce the reader to the following notions:
>
> **premiss**
> **conclusion**
> **syllogism**
> **the dictum de omni et nullo of Aristotle**
> **distributed term**
> **quantity (universal, particular)**
> **quality (affirmative, negative)**
>
> (2) Then in the chapter Synthetical Compendium he returns to the basic definitions and proceeds to build step by step the structure of Aristotelian logic, in a manner that is reminiscent of Euclid's development of geometry.

Now let us look at some details of his presentation. After all, this text was a, if not the, launching pad for the work of De Morgan and Boole.

In the chapter Analytical Outline Whately starts by saying that logic is the Art and Science of reasoning, and goes on to say what logic is not. Then he points out that in all the diversity of intellectual activities requiring reasoning, the processes of reasoning are really the same, and thus (pages 22–23):

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[2]The importance of language for reasoning is stated as follows by Boole in 1847 (page 5):

> The theory of Logic is thus intimately connected with that of language.

But in a postscript to the book he adds (page 81):

> Language is an instrument of Logic, but not an indispensable instrument.

It would be interesting to know how Whately reconciled the importance of language with the claim by Chrysippus that a good hunting dog has basic skills in reasoning:

> When running after a rabbit, the dog found that the path suddenly split in three directions. The dog sniffed the first path and found no scent; then it sniffed the second path and found no scent; then, without bothering to sniff the third path, it ran down that path.

> ... it could not but appear desirable to lay down some general rules of reasoning, applicable to all cases, by which a person might be enabled the more readily and clearly to state the grounds of his own conviction, or of his objection to the arguments of an opponent; instead of arguing at random, without any fixed and acknowledged principles to guide his procedure.

If one looks at an argument in detail he says (page 23):

> ... it will be found that every conclusion is deduced, in reality, from two other propositions; ...

and (page 24):

> An argument thus stated regularly and at full length, is called a Syllogism; which therefore is evidently not a peculiar *kind of argument*, but only a peculiar *form* of expression, in which every argument may be stated.
>
> When one of the premises is suppressed, (which for brevity's sake it usually is) the argument is called an Enthymeme.

After presenting some fallacious examples of syllogisms, e.g., on page 27:

> ... every rational agent is accountable; brutes are not rational agents; therefore they are not accountable ...

he gives a similar argument to demonstrate the fallacy:

> ... every horse is an animal; sheep are not horses; therefore they are not animals ...

and he says (pages 27–28):

> This mode of exposing a fallacy, by bringing forward a similar one whose conclusion is obviously absurd, is often, and very advantageously, resorted to in addressing those who are ignorant of Logical rules; ...

Then he gives two examples of correct syllogisms, one using (page 28):

> ... that whatever is said of the whole of a class, may be said of any thing comprehended in that class ...

and also (page 29)

> ... whatever is *denied* universally of any class may be denied of anything that is comprehended in that class.

These two statements comprise the **dictum de omni et nullo** of Aristotle, and it is asserted that this provides the basis for all correct reasoning (pages 29–30):

> On further examination it will be found, that all valid arguments whatever may be easily reduced to such a form as that of the foregoing syllogisms; and that consequently the principle on which they are constructed is the UNIVERSAL PRINCIPLE of Reasoning. ... it will be found that all the steps even of the longest and most complex train of reasoning, may be reduced into the above form.

But it is a mistake, he says, to think that Aristotle and other logicians intended that one should actually decompose everyday arguments into a detailed series of syllogisms.

Although some writers had ridiculed the principle of omni et nullo as being obvious, Whately praises Aristotle's dictum (page 32):

> ... it is the greatest triumph of philosophy to refer many, and seemingly very various, phenomena to one, or a very few, simple principles; ...

Then he explains the advantage of using symbols for the terms in a proposition, e.g., as in "All A is B", when analyzing an argument (page 35):

> ... to trace more distinctly the different steps of the abstracting process, by which any particular argument may be brought into the most general form.

And he goes on to say that the use of symbols clarifies the connection between the premises and the conclusion.

He turns to the importance of the **distributed term**, meaning a term in a proposition that is referred to in its entirety. For example in "All A is B" the term B is not distributed as we do not know if A refers to all or just part of B. But in the proposition "No A is B" it is clear that one is speaking of the entirety of B. He explains that an important principle of Aristotelian logic says that the middle term (the term that appears in both premises but not in the conclusion of a syllogism) must be distributed in at least one of the premises if a syllogism is valid. The failure to have a middle term that is distributed in at least one of the premises is a common feature of fallacious arguments as Whately demonstrates by examples.

Next he talks about the **quantity** (**universal, particular**) and the **quality** (**affirmative, negative**) of a proposition.

After this pleasant and meandering survey of some highlights of deductive logic he is ready to turn to the detailed presentation.

In the chapter Synthetical Compendium, Part I.—Of the Operations of the Mind and of Terms, Whately starts at the supposed beginning, the workings of the mind when reasoning, and presents the **three operations of the mind** that we summarize in the following table:

Three Operations of the Mind	Definition	Expression in language
simple-apprehension	conception of any object in the mind	**term (extreme)**
judgement	pronouncement on agreement or difference of two conceptions	**proposition**
reasoning	proceeding from one judgement to another founded on it	**act of reasoning; argument**

Fig. **3** The Three Operations of the Mind

and then he turns to definitions, with examples, of the following words:

Syllogism: A regularly expressed argument of three propositions each with two terms.
Subject: The term of a proposition "being spoken of".
Predicate: "that which is said of [the subject]".
Copula: "is" or "is not"; the middle of a proposition.
Simple term: A single word capable of being used as a term.
Singular term: A term that stands for one individual.
Common term, or **Predicable:** A term that stands for several individuals.

Whately goes into the detailed classification of terms, with the following vocabulary from biology:

genus	**individual**
species	**infima**
difference	**summum**

One could say that *man* is a species of the genus *mammal*; or that the *mammal* is a species of the genus *animal*. In terms of extension, a species denotes a *subclass* of a genus. One has the following heirarchy:

individual — infima ... species — genus ... summum

Difference is the qualification that one needs to add to the genus to get the species, e.g., adding the difference *rational* to the genus *animal* gives the species *man*.

An **accident** is an extra property (beyond that specified by the species) that applies to some of the members of a species, e.g., *blue-eyed* is an accident of the species *men*. A **separable accident** is one that need not always apply to an individual in a species, e.g., being *hungry* is a separable accident of the species *men*. An **inseparable accident** always or never applies to an individual of the species, e.g., *being chinese* is an inseparable accident of the species *men*.

Whately goes on to point out the classification of a term is relative to the context, and gives the following example that we have put in table form (page 67):

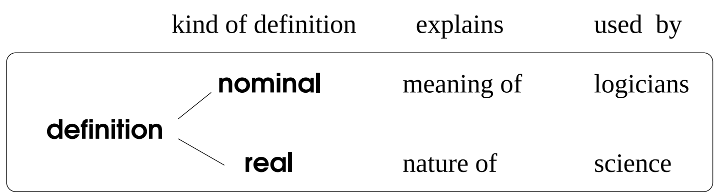
red in relation to	is classified as
pink	genus
rose	difference
blood	property
house	accident

He goes on to talk about the *common term* as an inadequate notion of the mind, and not an object in reality (as some authors claimed).

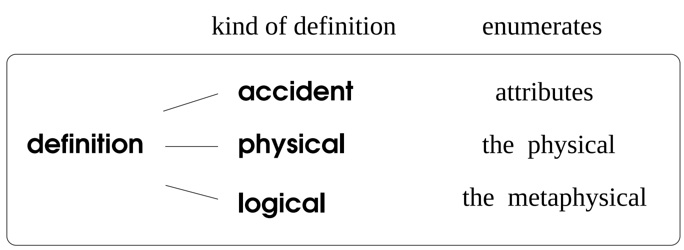
Then he discusses the **division** of a term. We would today prefer the terminology *subdivision* or *partition*. He gives the following rules for division (page 69):

> 1st. each of the Parts, or any of them short of *all*, must *contain less* ... than the thing divided. 2d. All the Parts together must be exactly equal to the thing divided; ... 3d. The Parts or Members must be *opposed; i.e.*, must not be contained in another: ...

Then there are the two different kinds of **definition** which we diagram as follows:



He says that in mathematics the two kinds of definitions coincide. Another partitioning of definition is given by:



Whately says logic deals with **nominal definitions**, and on page 74 he adds:

> It is scarcely credible how much confusion has arisen from the ignorance of these distinctions which has prevailed among logical writers.

The rules for definitions are that they must be:

1. adequate
2. plainer than the thing defined
3. a convenient number of appropriate words

In Part II.—Of Propositions. Whately says (pages 75–76):

> The second part of Logic treats of the *proposition*; which is, *"Judgement expressed in words."*
>
> A Proposition is defined logically *"a sentence indicative," i.e.*, affirming or denying; ... "Sentence" being the *genus*, and "Indicative" the *difference*, this definition expresses the whole essence; and it relates entirely to the *words* of a proposition. With regard to the *matter*, its property is, to be *true* or *false*.

Then he classifies sentences into **categorical** and **hypothetical**, depending on whether they are expressed either *absolutely*, or under an *hypothesis* . . .

Categorical propositions are divided into **pure** and **modal**. The first simply asserts that the subject does or does not agree with the predicate, but the second express in what *mode* (or manner) it agrees. We will be dealing only with the pure categorical statements. The **differentia** and **property** are defined, and again we present the data in a table:

	of a proposition
differentia	**affirms** or **denies**
property	**is true** or **is false**

The **quality** of the *expression* of a proposition is **affirmative** if the *copula* is affirmative, and it is **negative** if the *copula* is negative. The **quality** of the *matter* of a proposition is either **true** or **false**. The quality of the *expression* of a proposition is *essential*, that of the *matter* is *accidental*.

The **quantity** of a proposition is **universal** if the predicate applies to the whole of the subject, otherwise it is **particular**. In universal propositions the subject is said to be **distributed**, and in particular propositions the subject is **not distributed**.

This leads to the four kinds of (pure) categorical propositions (page 78) that we put into a chart with examples:

symbol	expresses	example
A	universal affirmative	All A is B
E	universal negative	No A is B
I	particular affirmative	Some A is B
O	particular negative	Some A is not B

Fig. **4** Pure Categorical Propositions

He notes that the **distribution** or **nondistribution** of the subject depends on the quantity of the proposition, whereas for the predicate the key is the quality of the proposition. Thus we have (using S for subject, P for predicate)

	affirmative	negative
universal	**S distributes**	**S and P distribute**
particular		**P distributes**

This is presented as two "practical" rules by Whately (page 80):

1st. All universal propositions (and no particular) distribute the *subject*.

2nd. All *negative* (and no affirmative) the predicate.

.

Any two different categorical propositions with the same subject and predicate are said to be **opposed**. Thus, given a subject and predicate, each pair of the A,E,I,O propositions are opposed, and although there are six pairs, it has been traditional to classify them into four kinds of **opposition** as summarized in the next table:

the pair	are
A,E	**contraries**
I,O	**subcontraries**
A,I	**subalterns**
E,O	**subalterns**
A,O	**contradictories**
E,I	**contradictories**

The word **matter** is not defined by Whately other than to say that it is the *substance*, but seems to refer to the meaning given to the proposition by a concrete choice of subject and predicate. Once the matter is fixed then the truth or falsity of a categorical proposition is determined.

Although the phrases **necessary matter**, **contingent matter**, and **impossible matter** are also not defined, a perusal of the examples suggests the following, given a concrete subject S and predicate P:

necessary matter: The universal affirmative (A) is true.
contingent matter: The particulars (I,O) are true.
impossible matter: The universal negative (E) is true.

This leads to the famous **square of opposition**, a diagram that summarizes the truth or falsity of the matter of the various forms A,E,I,O, depending on which of the above mentioned three cases hold:

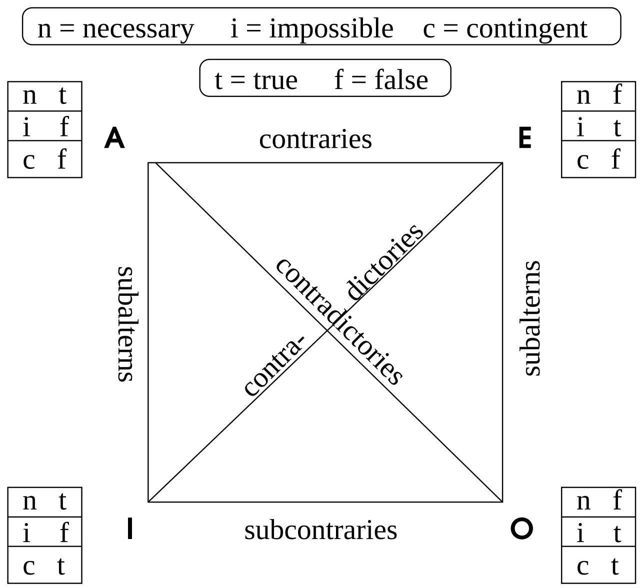


Fig. **5** The Square of Opposition

Thus, for example, we see that in the case of *impossible matter*, by looking at the i's in the four small boxes, we have A,I are false and E,O are true. (Whately used v, for *veritas*, rather than our t, for true.) Evidently there were maxims to describe every possible result in the above table, such as (pages 82–83):

> . . . Contraries will be *both false* in Contingent matter, but never *both true*: . . .
>
> ⋮
>
> . . . in the Subalterns, the truth of the particular (which is called the *subalternate*) follows from the truth of the universal (*subalternans*), and the falsity of the universal from the falsity of the particular: that Subalterns differ in *quantity alone*; Contraries, and also Subcontraries, in *quality alone*; Contradictories, in both: and hence, that if any proposition is known to be true, we infer that its Contradictory is false; if false, its Contradictory true, &c.

The "&c" seems to show that Whately does not wish to exhaust the reader with all the minute details.

The final subsection of the section on propositions deals with **conversion**, and we have on page 83:

> A proposition is said to be *converted* when its terms are *transposed*: when nothing more is done, this is called *simple* conversion. No conversion is employed for any logical purpose, unless it be *illative; i.e., when the truth of the Converse follows from* the truth of the Exposita, (or proposition given;) . . .

Conversions that switch from universal to particular are called conversion by **limitation**, or **Conversion per accidens**. Another form of conversion that he discusses is conversion by **negation**, also called conversion by **contraposition**. It was not included in his source, Aldrich, but was frequently used in Whately's time (as it is today). He summarizes the possibilities as:

> Thus, in one of these three ways, every proposition may be illatively converted: *viz. E, I, simply; A, O, by negation; A, E, limitation.*

The following table shows the details of these conversions:

premiss		kind of conversion		
		simple	**by limitation**	**contrapositive**
A	All A is B		Some B is A	All not-B is not A
E	No A is B	No B is A	Some B is not A	
I	Some A is B	Some B is A		
O	Some A is not B			Some not-B is A

Fig. **6** Conversions

In Part III.—Of Arguments he repeats the assertion that if an argument is fully written out, in so-called regular form, then one will be dealing with syllogisms. He repeats the dictum de omni et nullo of Aristotle in a more traditional way, but points out that one doesn't really want to reduce all one's arguments to this obvious form (pages 88–89):

> ... *whatever is predicated of a term distributed, whether affirmatively or negatively, may be predicated in like manner of everything contained under it.* ... This rule may be *ultimately* applied to all arguments: (and their validity ultimately rests on their conformity thereto) but it cannot be *directly and immediately* applied to *all* even of pure *categorical* syllogisms; for the sake of brevity, therefore, some other axioms are commonly applied in practice, to avoid the occasional tediousness of reducing all syllogisms to that form in which Aristotle's *dictum* is applicable.

He gives the two axioms or canons by which the validity of pure categorical syllogisms are to be proved. They are just the simplest rules for equality (page 90):

> ... first, *if two terms agree with one and the same third, they agree with each other:* secondly, *if one term agrees and another disagrees with one and the same third, these two disagree with each other.*

Then he launches into a detailed description of what really constitutes a syllogism, and what the nomenclature is to be. A **syllogism** will consist of exactly three propositions—two premises and one conclusion. The subject of the conclusion is called the **minor term** and the predicate of the conclusion is the **major term**. Each term of the conclusion will appear in exactly one of the premises. The premiss with the minor term is the **minor premiss**, that with the major term the **major premiss**. There will be a third term involved in each of the premises, called the **middle term**. Then we come to some important rules for valid syllogisms (page 91):

> *The middle term therefore must be distributed once, at least,* in the premises; ... and once is sufficient; ...
>
> ⋮
>
> *No term must be distributed in the conclusion which was not distributed in one of the premises;*
>
> ⋮
>
> *From* [two] *negative premises you can infer nothing.*
>
> ⋮
>
> *If one premiss be negative, the conclusion must be negative;*

According to the **Elements**, *these four rules are all one needs to determine if a (well-formed) syllogism is valid.* Actually Whately gives six rules, but two of them are just to check that one is indeed dealing with a syllogism.

Armed with this he turns to the complete classification of the valid syllogisms. Given a syllogism, the triple of letters describing the kind of major premiss, minor premiss, and conclusion, is called the **mood** of the syllogism. For example AAA is the mood of the syllogism

All Y is X	major premiss
All Z is Y	minor premiss
All Z is X	conclusion

There are clearly 64 (= $4 \times 4 \times 4$) moods.

The mood alone does not determine a syllogism, for one needs to know the location of the middle term in the premises. Following Whately, if we have Z for the subject and X for the predicate of the conclusion, and Y as the middle term, then one sees that there are four possible arrangements of the order of the terms, and each of these arrangements is called a **Figure**:

1st Fig.	2nd Fig	3rd Fig.	4th Fig	
Y X	X Y	Y X	X Y	major premiss
Z Y	Z Y	Y Z	Y Z	minor premiss
Z X	Z X	Z X	Z X	conclusion

Fig. **7** The Four Figures of Aristotelian Logic

Using the previous four rules one can then work through the various moods and discover that there are only 6 in each figure that are valid, i.e., there are *a total of 24 valid syllogisms.* Here is the complete list.

1st Figure	2nd Figure	3rd Figure	4th Figure
AAA	AEE	AAI	AAI
AAI	**AEO**	AII	AEE
AII	AOO	EAO	**AEO**
EAE	EAE	EIO	EAO
EAO	**EAO**	IAI	EIO
EIO	EIO	OAO	AII

Fig. **8** The Valid Syllogisms

Of these valid syllogisms the five in bold type were thrown out because they drew a particular conclusion when one could actually draw a stronger universal conclusion—in each bold face case just look at the syllogism immediately above it. This leaves *the 19 valid syllogisms recognized by the traditional Aristotelian logic.* Whately's book does not have them in such a compact form, but rather discusses them one at a time over several pages. One can find a brief presentation in the first chapter of De Morgan's 1847 book (pages 18–19).

Whately does not discuss the fact that Aristotle did not even bother to consider the 4th Figure (because it is so easily reduced to the 1st Figure). The introduction of the 4th Figure is credited to Galen (who worked on logic before Aristotle's works were translated into Latin in the fifth century A.D.) The 4th Figure did not often appear in works on logic before 1700.

The logicians of the Middle Ages attached great importance to the 19 valid syllogisms, and devised some remarkable lines to make it possible to memorize a great deal of information about them:

Barbara, Celarent, Darii, Ferioque prioris.
Cesare, Camaestres, Festino, Baroko, secundae.
Tertia Darapti, Disamis, Datisi, Felapton, Bokardo, Ferison habet.
Quarta insuper addit Bramantip, Camenes, Dimaris, Fesapo, Fresison.

De Morgan describes these lines in his 1847 book (page 150):

> . . . and the magic words by which they have been denoted for many centuries, words which I take to be more full of meaning than any that were ever made.

Whately wrote them as follows:

Fig. 1 bArbArA , cElArEnt, dArII, fErIOque prioris.
Fig. 2 cEsArE, cAmAEstrEs, fEstInO, bArOkO, secundae.
Fig. 3 tertia dArA ptI, dIsAmIs, dAtIsI, fElAptOn, bOkArdO, fErIsOn
habet: quarta insuper addit.
Fig. 4 brAmAntIp, cAmEnEs, dImArIs, fEsApO, frEsIsOn.

The words *prioris* indicates that the first line is for the 1st Figure, etc. The information was considered so important that Whately says (page 98):

> By a careful study of these mnemonic lines (which must be committed to memory) you will perceive . . .

And then he proceeds to explain how to decode them. Indeed they represent a remarkable amount of knowledge compression, describing how to reduce the syllogisms in the 2nd through 4th Figures to the 1st Figure by using conversion, etc. The first three vowels in each of the proper names describe the mood, e.g., in Ferioque we have the mood EIO. In all but the 1st Figure every consonant except t and n have special meanings to show how to carry out the reduction of the syllogism to the 1st Figure. Note that the names in the 1st Figure start with the consonants B, C, D, F, and that all the names in the other figures start with one of these four consonants. Here are the details:

- The name for a given syllogism will reduce to one in the 1st Figure with the same intial letter, e.g., the 3rd Figure Ferison will reduce to the 1st Figure Ferioque. There are two exceptions to this rule, and they are the only ones to have the letter k in the name, namely Baroko and Bokardo.
- A consonant s means that the proposition corresponding to the preceding vowel is to be simply converted.
- A consonant p means that the proposition corresponding to the preceding vowel is to be converted by limitation.
- The consonant m means that the premises are to be interchanged.

Let us apply these rules to the 3rd Figure Disamis, i.e., the 3rd Figure IAI syllogism, with the consonant over the arrow indicating which of the above transformations we are carrying out:

$$\begin{array}{ll} \mathrm{I} & \text{Some B is C}\\ \mathrm{A} & \text{All B is A}\\ \hline \mathrm{I} & \text{Some A is C}\end{array} \xrightarrow{s} \begin{array}{ll} \mathrm{I} & \text{Some C is B}\\ \mathrm{A} & \text{All B is A}\\ \hline \mathrm{I} & \text{Some A is C}\end{array} \xrightarrow{s} \begin{array}{ll} \mathrm{I} & \text{Some C is B}\\ \mathrm{A} & \text{All B is A}\\ \hline \mathrm{I} & \text{Some C is A}\end{array} \xrightarrow{m} \begin{array}{ll} \mathrm{A} & \text{All B is A}\\ \mathrm{I} & \text{Some C is B}\\ \hline \mathrm{I} & \text{Some C is A}\end{array}$$

and we have transformed Disamis into the promised Darii.

But the cases Baorko and Bokardo do not really admit such a reduction, so the logicians developed a method of reduction to absurdity to handle these cases. The method is to replace the premiss immediately preceding the k in the name (the minor premiss in the case of Baorko, the major premiss in the case of Bokardo) by the contradictory of the conclusion and derive a contradiction to the removed premiss.

Finally Whately looked at the **hypothetical propositions** and the **hypothetical syllogisms**, and gave several examples, noting that one could have many more hypotheses or disjuncts than the two in his examples. His examples were not given in display form as below, but as was usual throughout his text, they were linearly embedded in paragraphs of discussion.

$$\textbf{constructive conditional:}\quad \begin{array}{c} \text{If A is B then C is D}\\ \text{A is B}\\ \hline \text{C is D}\end{array}$$

destructive conditional:

$$\frac{\begin{array}{c}\text{If A is B then C is D}\\ \text{C is not D}\end{array}}{\text{A is not B}}$$

disjunctive syllogism:

$$\frac{\begin{array}{c}\text{A is B or C is D}\\ \text{A is not B}\end{array}}{\text{C is D}}$$

He noted that a disjunctive proposition could be converted into a conditional proposition, e.g., in this example one could change "A is B or C is D" to "If A is not B then C is D".

constructive dilemma:

$$\frac{\begin{array}{c}\text{If A is B then C is D; if X is Y then C is D}\\ \text{A is B or X is Y}\end{array}}{\text{C is D}}$$

destructive dilemma:

$$\frac{\begin{array}{c}\text{If A is B then C is D; if X is Y then E is F}\\ \text{C is not D or E is not F}\end{array}}{\text{A is not B or X is not Y}}$$

The final task of Whately, in this chapter that develops the classical logic, is to show that conditional syllogisms can be converted into categorical syllogisms (perhaps more than one is needed), and thus he is able to claim that the dictum de omni et nullo reigns supreme in logic.

He gives a concrete example, but we will simply take an abstract version of it, following Boole's idea of abbreviating the hypothetical "If A is B then C is D" into "If P then Q". The 'trick' is to take each categorical proposition P that is involved in the syllogism and replace it by a name of the form "The situation that P holds", turning a proposition into a class of circumstances in which it holds. If we let $\widehat{P}$ denote this class, etc., then the conditional proposition "If P then Q" becomes the categorical proposition "All $\widehat{P}$ is $\widehat{Q}$". This same idea would be used by Boole to apply his algebra of logic to the hypothetical propositions.

CHAPTER 2

George Peacock (1791–1858)

Peacock was an algebraist and did not work in logic. However it is widely believed that his attempt to give a proper foundation for algebra (in the period from 1830 to 1845) influenced Boole's view as to the soundness of using algebra even when one does not have an interpretation, or at least not one where the operations are always defined.

As a student at Cambridge George Peacock was a founding member of the Analytical Society, along with Babbage and Herschel. One of their main purposes was to change Cambridge mathematics from Newton's notation to the Leibniz notation used on the continent. For more than a hundred years England had been tied to the clumsy Newtonian notation of fluxions and fluents. Although the Analytical Society endured for only a couple of years, from 1812 to 1814, it set out a program that changed British mathematics. French mathematicians, especially Lacroix, Lagrange and Laplace, had written beautiful books that captivated the young English mathematicians. One of the first contributions of the Society was an abridged translation in 1816 of S.F. Lacroix's 3-volumes on calculus, Traité du calcul, written in 1797–1800. (Lacroix wrote an expanded version in the years 1810–1819). In 1817 Peacock was on the mathematics examining committee at Cambridge, and he introduced the continental notation, over considerable objections. Within two years the Newtonian notation had completely disappeared.

Peacock was a well read mathematician, and he recognized that there were serious gaps in the development of algebra. He undertook to give the subject a careful development, in the spirit of Euclid's Elements. In 1830 he published Treatise on Algebra, with the main innovation being his *Principle of the Permanence of Equivalent Forms.* In 1833 he presented a survey on the state of algebra to the Royal Irish Academy in which one sees that he is quite comfortable with the interpretation of algebra in what we now call the complex plane, although he regards it as only *an* interpretation. He is also familiar with Abel's work on the quintic, but he does not understand it.

1. Treatise on Algebra: Vol. 1 1842, Vol. 2 1845

Peacock's foundation (of 1830) for algebra did not catch on, and in the 1840s he decides to break the subject into two parts, Vol. 1 on Arithmetical Algebra (1842) and Vol. 2 on Symbolical Algebra (1845), to make his Principle clearer. Vol. 1 deals with the algebra of the positive numbers. Vol. 2 allows expressions that we now call complex numbers, and functions of complex variables. His intention was to do a comprehensive treatment of algebra, including all the work on curves and numerical methods of finding roots. However Vol. 2 concludes just with the elementary properties of the familiar transcendental functions a^x, $\log x$, $\sin x$, $\cos x$, etc., and with the well known formulas for the roots of polynomials of degree at most 4.[1]

[1]Unfortunately at the end of Vol. 2 he fumbles his treatment of the logarithm, concluding that $\log(-1)^2$ and $\log(1^2)$ do not have the same range of values (Vol. 2, p. 443):

> . . . it consequently appears that we are not authorized in inferring, as has sometimes been done, the identity of the logarithms of $(-a)^2$ and a^2, from the identity of the symbolical results to which they lead when the signs of affection or their recipients are suppressed.

(His mistake is simply that if $\log a$ represents the many valued logarithm of a, then $2 \log a$ will represent the many valued logarithm of a^2.) This in turn leads to his conclusion that negative numbers do not have logarithms (p. 444):

> . . . we may conclude, therefore, generally that there is no possible logarithm of a negative quantity.

Here is an outline of the topics covered in the two volumes:

Vol. 1. Arithmetical Algebra

- The algebra of polynomials and their quotients
- Roots of numbers
- Decimal expansions
- Continued fractions
- Ratio and proportion
- Linear and quadratic equations
- Permutations and combinations
- Binomial Theorem
- Linear diophantine equations
- Some elementary number theory

Vol. 2. Symbolical Algebra

- Principle of the Permanence of Equivalent Forms
- Roots
- General Binomial Theorem
- The nth roots of 1
- Trigonometry
- DeMoivre's Theorem
- Interpretation in the plane
- Series for exponentiation and log
- Series for $\sin x$ and $\cos x$
- Partial fractions
- Roots of equations of degree ≤ 4

For the most part these two volumes are a rather long winded treatment of the basic facts of algebra and trigonometry. Peacock's willingness to embrace the geometric interpretation that we call the complex plane put him ahead of many of his contemporaries.[2] But his attempt to found algebra on the Principle of the Permanence of Equivalent Forms is, in retrospect, simplistic and poorly presented. His interesting comments on this Principle are sparsely scattered in a sea of mundane arithmetic and algebraic computations.

The conditions under which his Principle apply are vague, and at the end of the Vol. 2 Peacock adds an extra restriction to exclude a counterexample found in Euler's work. His formulation of the Principle was given first in 1830, and then again in 1845, in Vol. 2, p. 59:

> *"Whatever algebraical forms are equivalent, when the symbols are general in form but specific in value, will be equivalent likewise when the symbols are general in value as well as in form."*

This says that if an equation holds on the positive numbers studied in arithmetic then it is considered true in Symbolical Algebra. He goes on to say:

> It will follow from this principle, that all the results of Arithmetical Algebra will be results likewise of Symbolical Algebra: and the discovery of equivalent forms in the former science, possessing the requisite conditions, will be not only their discovery in the latter, but the *only* authority for their existence: for there are no definitions of the operations in Symbolical algebra, by which such equivalent forms can be determined.

Symbolical Algebra takes place on the syntactic level, and admits of at least two interesting interpretations, one in arithmetic, and the other using oriented line segments in the plane. An interpretation need not interpret every expression in Symbolical Algebra, but insofar as it does interpret expressions the results (equations) should be correct for that interpretation.

The two most trumpeted applications of his Principle are the following:

- The properties of a^x, where x need not be a positive integer, are derived from the arithmetical properties of a^n where n is a positive integer, namely $a^m \cdot a^n = a^{m+n}$ and $(a^m)^n = a^{mn}$.
- The Binomial Theorem

$$(1+x)^n \;=\; 1 + nx + \frac{n\cdot(n-1)}{1\cdot 2}x^2 + \cdots\,,$$

[2]For example, in late 1847 we find Cayley corresponding with Boole saying:

> I wonder we should never have stumbled in our previous correspondence on the subject of my *utter disbelief* of the received "English" theory of the geometrical interpretation of $\sqrt{-1}$. I would much more easily admit witchcraft on the philosopher's stone.

Boole simply regarded $\sqrt{-1}$ as *uninterpretable*.

where n need not be a positive integer, is obtained by his Principle from the corresponding result of Arithmetic, where n is a positive integer.

Note that Peacock's Principle is supposed to justify the transition from a finite sum to an infinite series, without any need to discuss convergence. He chastises Euler for not having adopted this Principle since he finds that it is the only way to justify the Binomial Theorem (Vol. 2, p. 452):

> Euler had drawn the same conclusion, nearly in the same manner, in his celebrated proof of the series for $(1+x)^m$ [a footnote gives the reference: Acta Petropol., 1774], though he at the same time denied the universal application of a principle equivalent to that of the permanence of equivalent forms, which alone could make it valid ...

The reader is faced with the question "What precisely does Peacock mean?" in many places. Here are some examples of such questions:

- Vol. 1 is supposed to be the algebra for the numbers of common arithmetic. At first such numbers are described as those which can be expressed using the digits 0, ... ,9 (Vol. 1, p. 1):

  > 1. Arithmetical Algebra is the science which results from the use of symbols and signs to denote the numbers and the operations to which they may be subjected; these numbers, or their representatives, and the operations upon them, being used with the same sense and with the same limitations as in common arithmetic.
  >
  > ... Those numbers which are actually assigned and given, are expressed by means of the nine digits and zero, by the aid of the artifices of ordinary arithmetical notation ...

  Certainly this includes the positive numbers (it is clear that he does not include 0 or any negative numbers), but soon fractions are declared to be numbers. There is strong evidence that the **positive rationals** are what he means by the numbers of Arithmetic (Vol. 1, p. 273):

  > 416. In the solution of the preceding problems we have generally used the word number in its largest sense, as signifying fractional as well as whole numbers ...

  But what is one to make of his treatment of roots of numbers? (Vol. 1, p. 130):

  > 214. The square root of a number is that number, whether expressed by a finite series of digits or not, which multiplied by itself will produce the primitive number ... the square root of 10 is interminable ...

  Vol. 1 also treats infinite decimals and continued fractions, and notes that every geometric magnitude can be expressed by a decimal (Vol. 1, p. 92):

  > 169. It thus appears that decimals, either definite or indefinite, are competent to express the values, not merely of *commensurable* magnitudes, which are multiples of some assignable subordinate unit, but also of such as are *incommensurable* ...

  He says that numbers are not continuous, whereas geometric magnitudes are continuous, and again seems to emphasize that numbers are rational (Vol. 1, p. 161):

  > 278. Numbers, whether rational or surd, are essentially discontinuous, and in strictness of language, are incapable of expressing as symbols the properties of continuous magnitude ... consequently no number can become the absolute representative of an incommensurable magnitude.

  In Vol. 2 we encounter e, π, and transcendental functions. What precisely does Peacock mean by the numbers of Arithmetic? Is $\sqrt{x}$ or e^x such an arithmetical number when x is such a number? etc.
- What are the operations permitted in his Arithmetic? In Vol. 2 he shows, by invoking his Principle, that in Symbolical Algebra $a^{4/5}$ is the fifth root of the fourth power of a. Does this make $a^{4/5}$ an operation of Arithmetic?
- What kinds of expressions are allowed in making equations? Some of the functions he uses, like $\sqrt[n]{x}$ and $\log x$, are multivalued in his Symbolical Algebra. Although $x = \sqrt{x^2}$ is true in

the positive numbers, one cannot apply his Principle here and conclude that this holds in Symbolical Algebra (for then one would have the favorite paradox of $1 = -1$).

- Since the operation of subtraction is *partial* in his Arithmetic (of positive numbers), what does Peacock mean when he says an equation holds in Arithmetic? Does it have to hold at least for all positive integers?
- Even if an equation holds, Peacock may not permit the application of his Principle to it on the grounds that it has not been derived by proper means. Euler's example of a series that takes the value x when x is a positive integer, but not otherwise, is rejected by Peacock on these grounds (Vol. 2, p. 452):

  > ... he [Euler] produced, as a striking exception to [the Principle's] truth, the very remarkable series
  > $$\frac{1-a^m}{1-a} + \frac{(1-a^m)(1-a^{m-1})}{1-a^2} + \frac{(1-a^m)(1-a^{m-1})(1-a^{m-2})}{1-a^3} + \cdots$$
  > whose sum is m, when m is a whole number, but not so for other values.
  > A little consideration, however, will be sufficient to shew that the principle of the permanence of equivalent forms is not applicable to such a case: for if m be a whole number, as in Arithmetical Algebra, the connection between m and its equivalent series in the identical equation
  > $$m = \frac{1-a^m}{1-a} + \frac{(1-a^m)(1-a^{m-1})}{1-a^2} + \frac{(1-a^m)(1-a^{m-1})(1-a^{m-2})}{1-a^3} + \cdots$$
  > is not given, or, in other words, there is no statement or definition of the operation, by which we pass from m, on one side of the sign $=$, to a series under the specified form on the other, and there is consequently no basis for the extension of the conclusion to all values of the symbols, either by the principle of the permanence of equivalent forms or by any other: it is only when the results, which are general in form, but specific in value, are derived by processes which are definable and recognized, that they become proper subjects for the application of this principle.

Except for possibly giving Boole a green light to proceed with his algebra of logic, Peacock's work seems to have had little lasting influence on mathematics. His fundamental principle was too vague to offer a foundation—any reasonable attempt to make it precise leads to results that contradict our basic structure of complex numbers, and functions of complex numbers.

CHAPTER 3

Augustus De Morgan (1806–1871)

Augustus De Morgan was born in India, the fifth child of a Colonel in the British Army. His father was the third generation of a family of army officers working with the East India company, but Augustus had a defective right eye and could not follow in this family tradition. He was brought to England in his infancy, attended some ordinary private schools, and acquired a fine classical education. At 16 he entered Trinity College, Cambridge, and soon took up mathematics and philosophy as his main interests. As a student he was involved with the movement to replace the Newtonian notation in British analysis with the Leibnizian notation used on the Continent. In 1827 he obtained his B.A., but was not permitted to hold a fellowship or pursue the M.A. as he was not willing to subscribe to the required religious affirmations. He always considered himself a 'Christian unattached'.

Fortunately the newly founded University of London was nondenominational, and De Morgan, at the age of 22, competed successfully (against 33 other candidates) to win the mathematical professorship. He would spend most of the rest of his life there.[1] He was regarded as a brilliant teacher, and a great believer in the importance of a strong foundation in logic when pursuing mathematics. His students included Sylvester, Lady Lovelace, and Jevons.

De Morgan's work in logic[2] can be summarized as a single-minded quest to improve the syllogism as the main instrument of reasoning, keeping in mind that the truths embodied in the accepted inferences of Aristotelian logic should be preserved. At the same time that he is anchoring his logical investigations on the syllogism he will occasionally question the adequacy of the underlying propositions to faithfully capture the assertions of the common language. And on one occasion he even questions the adequacy of the syllogism itself to capture inference.

But De Morgan does not question the fact that logic is concerned with propositions $\varphi(\mathrm{X},\mathrm{Y})$ that relate X and Y. Nor does he challenge the form of the categorical proposition

quantifier	subject	copula	predicate

except to allow certain conjunctions of these propositions (called complex propositions). Nor does he depart from the form of the syllogism

$$\begin{array}{l} \mathsf{premise(X,Y)} \\ \mathsf{premise(Y,Z)} \\ \hline \mathsf{conclusion(X,Z).} \end{array}$$

The main thrust of his work is to extend the possibilities for the components of the categorical proposition, combining this with developing compact symbolic notation for propositions and syllogisms. The following diagram shows De Morgan's main ideas for modifying the categorical proposition:

[1] In 1831 he resigned because of the way the governing body was treating the faculty, and lived 'in the wilderness' for five years. But after five years his successor at the University of London died in an accident and De Morgan was unanimously recalled.

[2] Including his book he published about 800 pages on logic. This work took place during the years 1839 to 1863, but mainly mainly between 1846 and 1850.

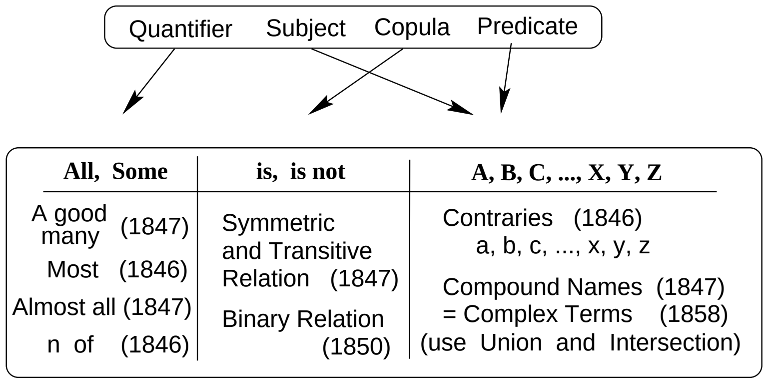


Fig. **9** Elements of the Categorical Proposition
and De Morgan's Ideas for Revision

His modifications of the quantifier simply provide additional categorical propositions. Only the definite quantifier 'n of' plays a significant role in his work.

To improve the copula De Morgan uses a binary relation. At first (1847) he uses a symmetric and transitive relation, but the significant step occurs in 1850 when he extends this to an arbitrary binary relation. His focus on the syllogism leads him to consider contraries, converses, and compositions of binary relations.

His first extension of the Aristotelian term is to include names for contraries, based on having a universe of discourse. Although De Morgan questions many aspects of Aristotelian logic, this is the one place where he makes a change that directly conflicts with the writings of Aristotle—consequently he devotes considerable space to justifying his decision to use names for contraries.

Later he introduces compound names based on forming unions and intersections. Compound names are introduced for the sole purpose of converting other forms of syllogisms, like dilemmas, into categorical syllogisms. He wants to make sure that there are names for the contraries of compound names, and this leads him to state the famous De Morgan laws.

Before discussing details of De Morgan's work we want to describe the conflict between De Morgan and Hamilton that had, as its main consequence, the fact that Boole wrote a book on logic.

In late 1846 De Morgan exchanged correspondence with the prominent and influential logician Sir William Hamilton,[3] Professor of Philosophy at the University of Edinburgh, an exchange that led to some very hard feelings, and on Hamilton's part some very harsh words. In March of 1847 Hamilton wrote to De Morgan charging him with plagiarism, with stealing Hamilton's discovery of *the quantification of the predicate*. For example, 'All X is all Y' in Hamilton's system was equivalent to a statement of the form 'D' in De Morgan's system, that is, to the complex proposition 'All X is Y and All Y is X'. De Morgan was incensed at such a charge, and demanded a retraction. A pseudo retraction was given by Hamilton, but this was far from enough to quiet the situation. The dispute quickly spilled into a public exchange in the journal Athenaeum, and continued until about 1852 when De Morgan ceased to engage in the conflict.[4] But De Morgan seems to have cherished the old battle, and brought it up repeatedly after Hamilton's death.

When De Morgan's book was published in November of 1847 there was a reference to the dispute in the preface, and a substantial appendix devoted to quoting Hamilton and asking the

[3]Not to be confused with Sir William Rowan Hamilton of Dublin, of quaternion fame, who was a friend of De Morgan.

[4]This may have been partly due to Hamilton's poor health. He had been partially paralyzed since 1846, and died in 1856.

reader to decide if Hamilton's charge was justified. Here are some samples of Sir Hamilton's prose taken from De Morgan's Appendix (pages 355, 363):

> In reply to your letter in the last number of the *Athenaeum*:—you were not wrong to abandon your promise "of trying the strength of my position;" for never was there a weaker pretension than that, by you, so suicidially maintained.
>
> ⋮
>
> I disregard your misrepresentation that "I avenge myself for the retraction of my aspersion on your integrity by my copious and slashing criticisms on your intellect." When your (excusable) irritation has subsided, you will see that I *could only* secure you from a verdict of plagiarism by bringing you in as suffering under an illusion.

Boole, a school teacher correspondent of De Morgan, became so intrigued in the spring of 1847 by this controversy that he was led to recall some of his own ideas on logic from his teenage years. And soon he too was writing a book on the subject. No doubt it was because of the spat with Hamilton that De Morgan wrote to Boole in the summer of 1847 asking him *not* to send De Morgan a draft of the logic book that Boole was working on until De Morgan had finished his own. In the introduction to Boole's book, also published in November of 1847, we find the following quote attributed to Hamilton:

> [The pursuits of the mathematician] have not only not trained him to that acute scent, to that delicate, almost instinctive, tact which, in the twilight of probability, the search and discrimination of its finer facts demand; they have gone to cloud his vision, to indurate his touch, to all but the blazing light, the iron chain of demonstration, and left him out of the narrow confines of his science, to a passive *credulity* in any premises, or to an absolute *incredulity* in all.

Augustus De Morgan was a transitional logician, educated in the traditional logic that was solidly based on the Aristotelian syllogism, active in the reform of logic, and supportive of the new developments (of Boole) in logic. He is not particularly noted for his mathematical achievements, but he was a prolific writer, his literary and mathematical output (books, papers, reviews, pamphlets, etc.) being possibly the greatest among the mathematicians of his time.

He married Sophia Elizabeth Frend, the daughter of a fellow mathematician, and they had seven children. De Morgan described himself as the most 'ungregarious animal living', and refused to seek election to the Royal Society, declined the offer of an honorary doctorate from Edinburgh, saying 'he did not feel like an LL.D.', refused his former students' request to have his portrait painted for presentation to the University College, and prided himself on the fact that he never voted, reflecting his general attitude toward politics. He resigned from the University for the second and last time in 1866.[5]

1. Formal Logic (1847)

De Morgan's book, Formal Logic, published in November of 1847, is 392 pages long, but only the first 170 pages deal with deductive logic. His book is filled with ideas for reworking classical logic, the key ones dealing with

- A universe of discourse
- Names for contraries (to augment the SYMBOLS)
- An abstract copula
- Disjunctions, conjunctions, and negations of propositions
- Abbreviated notation for propositions
- Transformations of propositions
- Structure and notation for syllogisms
- Graphical aids

[5] This was precipitated by the failure of the university to appoint a certain candidate to the chair of mental philosophy (because of his prominent Unitarian status).

- Compound names using union and intersection of SYMBOLS
- Semantically loaded readings of forms

In Chapter I, First Notions, the first 25 pages of his book, De Morgan gives a compact treatment of the traditional logic, up to and including the 19 traditional syllogisms. It is actually a reprint of a tract that he had published in 1839 called First Notions in Logic (preparatory to the study of geometry).

Using the notation (F) for the **denial**[6] of a given form F, he makes a table to show the consequences of each of the traditional forms as well as their denials (page 6):

From A	follow	(E), I, (O)	From (A)	follow	O
From E	···	(A), (I), O	From (E)	···	I
From I	···	(E)	From (I)	···	(A), E, O
From O	···	(A)	From (O)	···	A, (E), I

Fig. **10** Table of Immediate Inferences

Such detailed tables, largely left to the reader to check, are a frequent feature of the first half of this book. After discussion of immediate inferences De Morgan is ready for more complicated arguments and gives a definition of the syllogism (pages 7–8):

> Having thus discussed the principal points connected with the simple assertion, I pass to the manner of making two assertions give a third. Every instance of this is called a *syllogism*, the two assertions which form the basis of the third are called *premises*, and the third is the *conclusion*.

He really means that his syllogism will closely resemble the traditional syllogism—as we stated earlier there are two premises and a conclusion, each proposition will relate two terms, with a total of three terms in all, and no two of the three propositions involved will mention the same two terms.

De Morgan summarizes the essence of reasoning, a summary that echoes precisely the two canons presented in Whately:

> The paragraph preceding contains the essential parts of all inference, which consists in comparing two things with a third, and finding from their agreement or difference with that third, their agreement or difference with one another. ... Simple *identity* or *non-identity* is the ultimate state to which every assertion may be reduced; and we shall, therefore, first ask, from what identities, &c., can other identities, &c., be produced?

1.1. Objects, Ideas, Names. De Morgan has three categories of things that one can refer to in a proposition, namely **objects**, **ideas**, and **names**. Objects are in the external real world. Ideas are representations, in the mind, of objects or other things. And names are labels we provide for ideas. In a proposition he says that the terms should both refer to objects, or both refer to ideas, or both refer to names,[7] and that the general theory developed (that is, the immediate inferences and syllogisms) does not depend on which of these instances one has in mind. We will simplify this part of his system by simply saying that the terms in a proposition refer to *classes of things*.

1.2. Names for Contraries. His first change to Aristotelian logic is to extend the classes for which there are names to include *complements* of named classes. On page 38 he introduces his upper case/lower case notation for contraries, the contrary of A being a.[8] As this is the one

[6] Unfortunately the invention of a notation for the denial of a proposition disappears after this brief chapter. In subsequent work he uses ' ... denies F'. Thus a promising beginning to Boolean combinations of propositions is not pursued.

[7] In 1850 he challenges this requirement ([**5**], pages 59–60).

[8] This convention of using upper case roman font and lower case italic font will be adopted by Jevons, with credit to De Morgan. But, in reality, after this introduction De Morgan uses the roman font for both upper case and lower case symbols throughout the rest of his book. And in his several papers on the syllogism he uses the italic font for both upper and lower case symbols.

alteration of De Morgan that the writings of Aristotle explicitly disagree with he gives reasons for this decision (pages 40–41):

> ... make it desirable to include in a formal treatise the most complete consideration of all propositions, with reference not only to their terms, but also to the contraries of those terms.
>
> ⋮
>
> It may be objected that the introduction of terms which are merely negations of the positive ideas contained in other terms is a species of fiction. I answer, that, first, the fiction, if it be a fiction, exists in language, and produces its effects: nor will it easily be proved more fictitious than the invention of sounds to stand for things. But, secondly, there is a much more effective answer, which will require a little development.
>
> When writers on logic, up to the present time, use such contraries as man and not-man, they mean by the alternative, man and everything else. There can be little effective meaning, and no use, in a classification which, because they are not-men, includes in one word, *not-man*, a planet and a pin, a rock and a featherbed, bodies and ideas, wishes and things wished for. But if we remember that in many, perhaps most, propositions, the range of thought is much less extensive than the whole universe, commonly so called, we begin to find that the whole extent of the subject of discussion is, for the purpose of discussion, what I have called a *universe*, that is to say, a range of ideas which is either expressed or understood as containing the whole matter under consideration. In such universes, contraries are very common: that is, terms each of which excludes every case of the other, while both together contain the whole.

After pointing out in such cases that a term and its contrary can both be of interest he says:

> Accordingly, of two contraries, neither must be considered as *only* the negation of the other: except when the universe in question is so wide, and the positive term so limited, that the things contained under the contrary name have nothing but the negative quality in common.

And later De Morgan says that Aristotle's opposition to names for contraries was likely because he had not considered a limited universe (page 128):

> Aristotle will have no contrary terms: not-man, he says, is not the name of anything. He afterwards calls it an indefinite or *aorist* name, because, as he asserts, it is both the name of existing and non-existing things ... I think, however, that the exclusion was probably dictated by the want of a definite notion of the extent of the field of argument, which I have called the *universe* of the propositions. Adopt such a definite notion, and, as sufficiently shown, there is no more reason to attach the mere idea of negation to the contrary, than to the direct term.

1.3. Generalizing the Copula. After introducing names for contraries De Morgan turns to analyze the copula. He differs from the usual literature by having a term refer to instances rather than the whole class. 'X is Y' does not assert a relation between two classes X and Y, but rather the letters X and Y refer to *instances* (i.e., elements) of the respective classes. Thus 'X is Y' really means 'member-of-X is member-of-Y'. (In the traditional logic the relation 'is' is that of identity.) With this one sees that 'All X is Y' means 'All members of X are members of Y'. Hence we see De Morgan writing 'All Xs are Ys'.

He distinguishes his interpretation of 'X is Y' from the usual in the following passage (page 48):

> [logicians] would rather draw their language from the idea of two areas, one of which is larger than the other, than from two collections of indivisible units, one of which is in number more than the other.

We tend to think of one area being contained in the other precisely when every point of the first is a point of the second. Perhaps a better example would have been to take two simple closed curves in the plane with one inside the other. And his statement about 'indivisible units, one of which is in number more than the other' has to be understood as 'subset of'. His emphasis on the 'number' of indivisible units is surely misleading. He goes on to say:

> I shall take particular care to use numerical language, as distinguished from magnitudal, throughout this work, introducing of course, the plurals Xs, Ys, Zs, &c.

He means that he will emphasize the role of the elements of the classes. We will assume that the following table expresses De Morgan's understanding of categorical propositions in a more modern syntax:

proposition	modern version
All X is Y	$(\forall\alpha \in \mathrm{X})(\exists\beta \in \mathrm{Y})\,(\alpha \text{ is } \beta)$
No X is Y	$(\forall\alpha \in \mathrm{X})(\forall\beta \in \mathrm{Y})\,(\alpha \text{ is not } \beta)$
Some X is Y	$(\exists\alpha \in \mathrm{X})(\exists\beta \in \mathrm{Y})\,(\alpha \text{ is } \beta)$
Some X is not Y	$(\exists\alpha \in \mathrm{X})(\forall\beta \in \mathrm{Y})\,(\alpha \text{ is not } \beta)$

De Morgan turns to the question of which properties of 'is' are actually needed for inference, and says there are just three (page 50):

(1) 'X is Y' implies 'Y is X'
(2) 'X is Y' and 'X is Z' imply 'Y is Z'
(3) 'X is not Y' is the contradictory of 'X is Y'

These three properties hold precisely for the **symmetric** and **transitive** binary relations R, where the third property just means we take 'X is not Y' to be *not*(XRY). De Morgan will use (2) as though it were the transitive law. No doubt the wording of (2) was inspired by an axiom from Euclid, namely that 'things equal to the same are equal to one another'.

The necessity of the first property follows from preserving the Aristotelian inference called conversion of a particular—'Some X is Y' implies 'Some Y is X'—by applying this inference when X and Y denote singletons (De Morgan calls such propositions **doubly singular**). Likewise from the 3rd Figure AAI one has property (2). (Using the 1st Figure AAA gives the transitive property.) Thus any copula that preserves the Aristotelian inferences must satisfy (1)–(3).

He says any relation satisfying these three properties will serve equally well as a copula (page 51):

> ... we have power to invent new meanings for all the forms of inference, in every way in which we have power to make meanings of *is* and *is not* which satisfy the above conditions.

Indeed his three properties are also sufficient to preserve the inferences of Aristotelian logic, but he leaves the justification of this fact entirely to the reader. As an example of new meanings he gives the following (page 51):

> For instance, let X, Y, Z, each be the symbol attached to every instance of a class of *material* objects, let *is* placed between two, as in "X is Y" mean that the two are tied together, say by a cord, and let X be considered as tied to Z when it is tied to Y which is tied to Z, &c. There is no syllogism but what remains true under these meanings.

Then he goes on to say that for some syllogisms one does not need all three properties of 'is' mentioned above:

> Thus in the most common case of all, "Every A is B, every B is C, therefore every A is C," of all the three conditions only the second is wanted to secure the validity of this case.

Here De Morgan is careless as the second property does not express transitivity. He needs the first property (symmetry) along with the second property to obtain the transitive property of 'is'. He will make a similar error later on the same page with an example that is transitive but does not satisfy the second condition (even though he claims it does).

His claim that the three properties are sufficient for all the (valid) forms of inference is correct if one only considers Aristotelian inferences, but he will soon develop his logic that permits names for contraries in categorical propositions (see §1.4). Although De Morgan does not say what constitutes an acceptable copula for his new system, we assume that it must yield the valid inferences obtained when using the copula 'is identical to'. Presumably in Formal Logic De Morgan believes that the properties (1)–(3) of the copula will continue to be sufficient in his new logic.[9]

In his 1850 paper On the Syllogism II, De Morgan drops the third property from his requirements of the copula, and changes the second property to the transitive property ([**5**], page 51). Furthermore he realizes that a symmetric and transitive copula will not yield all the valid inferences of his new system, so he adds (without any explanation) the new requirement:[10]

'X is Y' or 'X is y' should hold for any X.

We take this to mean

$$(\forall\alpha\in \mathrm{X})\Big[(\exists\beta\in \mathrm{Y})(\alpha \text{ is } \beta)\ \ \text{or}\ \ (\exists\beta\in \mathrm{y})(\alpha \text{ is } \beta)\Big].$$

This can be replaced by the simpler condition

$$(\forall\alpha\in \mathrm{U})(\exists\beta\in \mathrm{U})(\alpha \text{ is } \beta),$$

U being the universe.

De Morgan never realizes that adding his new condition to the symmetric and transitive properties is equivalent to adding the **reflexive property**[11] $(\forall\alpha\in \mathrm{U})(\alpha \text{ is } \alpha)$, or, in De Morgan's mode of expression, 'All X is X'. Thus he is essentially claiming by 1850 that **equivalence relations** are the appropriate abstract copulas for his extension of categorical logic.

Unfortunately De Morgan fails to see that the only copula that gives the valid inferences of his new system is the original relation of identity. Indeed, if one just adds to the Aristotelian system the fact that each of the two propositions

'No X is Y' and 'All X is y'

can be inferred from the other then the copula is forced to be the identity relation.[12] We show this as follows.

The symmetric and transitive properties follow from preserving the Aristotelian inferences. We derive the reflexive property from the above equivalence by first assuming $\alpha\in \mathrm{U}$ is not related by the copula to any $\beta\in \mathrm{U}$. Let $\mathrm{X}=\mathrm{y}=\{\alpha\}$. Then 'No X is Y' holds, so we can infer 'All X is y' holds, and this yields 'α is α'. Thus $(\forall\alpha\in \mathrm{U})(\exists\beta\in \mathrm{U})\,(\alpha \text{ is } \beta)$ must hold. Now given α choose β such that 'α is β' holds. By symmetry 'β is α' holds, and then by transitivity 'α is α' holds. Thus the copula must be reflexive.

Next we show that the copula must be the identity relation. Suppose $\alpha\neq\beta$, and let $\mathrm{X}=\{\alpha\}$ and $\mathrm{Y}=\{\beta\}$. Then 'All X is y' holds (by the reflexive property), so 'No X is Y' holds. Thus 'α is not β' holds. This finishes our proof.

In the book Formal Logic De Morgan does not introduce a symbol for the underlying binary relation 'is'. Even in the most popular case of equality (=) he does not use a symbol until 1860.[13]

[9]He introduces the idea of names for contraries on page 37, before discussing the abstract copula—but the valid inferences, using names for contraries, are presented after this discussion. And there is no further discussion in Formal Logic of the nature of the abstract copula.

[10]His phrasing of the condition is as follows, where '—' is the copula ([**5**], page 52):

> When contraries are introduced, the copula condition further required is that either X—Y or X—y should hold for any X.

[11]This awkwardness in dealing with the reflexive property will manifest itself later in the work of Jevons.

[12]One easily sees that his example with objects tied together could satisfy only one of these two simple propositions—if no object in X is tied to any object in Y then one cannot conclude that every object in X is tied to some object in the complement of Y.

[13]He does use the usual equality symbol (=) in the sense of 'is defined as' or, with propositions, as 'equivalent to'. But unfortunately he often uses it in the sense of 'implies', for example, in his version (on page 88) of the AAA syllogism 'X)Y + Y)Z = X)Z'.

In his 1858 paper On the Syllogism III, page 87, he says that the notation A+B=C for 'A and B imply C' is

1.4. Simple Propositions. By admitting contrary terms De Morgan quadruples the number of categorical propositions. This gives his **simple propositions**, namely one has (for two SYMBOLS X,Y) the following 32 possibilities:

$$\begin{bmatrix}\text{All}\\ \text{Some}\end{bmatrix} \begin{bmatrix}\text{X}\\ \text{x}\end{bmatrix} \begin{bmatrix}\text{is}\\ \text{is not}\end{bmatrix} \begin{bmatrix}\text{Y}\\ \text{y}\end{bmatrix}$$

$$\begin{bmatrix}\text{All}\\ \text{Some}\end{bmatrix} \begin{bmatrix}\text{Y}\\ \text{y}\end{bmatrix} \begin{bmatrix}\text{is}\\ \text{is not}\end{bmatrix} \begin{bmatrix}\text{X}\\ \text{x}\end{bmatrix}$$

Fig. **11** Simple Propositions

This notation means that one can choose either of the possibilities in each of the bracketed items. Here the lower case/upper case letters are contraries, that is, x means not-X in De Morgan's notation.

One of the chief goals of De Morgan's book is to determine the valid simple syllogisms, and to give a complete set of rules for this purpose. With 32 ways to make a simple proposition this gives a total of $32^3 = 32,768$ simple syllogisms to consider. The first step towards this classification is to determine which of the simple propositions are semantically the same, and to select just one representative from each equivalence class.[14]

Let us say that two propositions are **equivalent** if each can be inferred from the other, under the restricted semantics—De Morgan says 'is the same as' instead of 'is equivalent to'. A proposition FXY is **convertible** if FXY is equivalent to FYX. Likewise for FXy, etc. Otherwise a proposition is **inconvertible** (see page 59).

There are eight equivalence classes among the 32 propositions (with reference to the SYMBOLS XY)—they are given in the columns below:

Universal Propositions		Particular Propositions	
All X is Y	All X is y	**Some X is Y**	Some X is y
All y is x	All Y is x	Some Y is X	Some y is X
No X is y	**No X is Y**	Some X is not y	**Some X is not Y**
No y is X	No Y is X	Some Y is not x	Some y is not x
All x is Y	**All x is y**	Some x is Y	**Some x is y**
All y is X	All Y is X	Some Y is x	Some y is x
No x is y	No x is Y	**Some x is not y**	Some x is not Y
No y is x	No Y is x	Some Y is not X	Some y is not X

Fig. **12** Equivalence Classes of Simple Propositions

. . . seriously objectionable, and must be discontinued.

Indeed he says it is faulty in two points, and that he should have written 'AB < C'.

Finally, in his Syllabus [**5**] of 1860, in paragraph 57 (pages 164–165), he introduces the notation '‖' to express 'is the identical of', for use with terms. Also he uses ‖ in a few relational identities.

[14] This procedure differs from the approach to Aristotelian syllogisms, where no such reduction is used for the basic classification. Of course semantic equivalence is used (conversion, etc.) to help show that one can reduce the valid syllogisms to the first figure. But De Morgan uses semantic equivalence reduction up front with simple propositions to reduce the huge number of cases to consider.

The representatives are those in bold type. Note that they are distinguished by the fact that either both X and Y occur, in that order; or their contraries x and y occur, in that order. De Morgan introduces eight **forms** for his simple propositions, namely

$$\mathrm{A}_{\mathsf{I}} \quad \mathrm{A}^{\mathsf{I}} \quad \mathrm{E}_{\mathsf{I}} \quad \mathrm{E}^{\mathsf{I}} \quad \mathrm{I}_{\mathsf{I}} \quad \mathrm{I}^{\mathsf{I}} \quad \mathrm{O}_{\mathsf{I}} \quad \mathrm{O}^{\mathsf{I}}$$

He says A_{I} is to be read 'sub-A' and A^{I} as 'super-A', etc. He refers to the 'sub' and 'super' as the **prepositions** of the form.

Here is the list of the **simple propositions** used by De Morgan (with reference to the SYMBOLS XY) and his **symbolic abbreviations**:

Proposition	Abbrev.	Expresses	Modern Symbolic Rendering
A_{I}XY	X)Y	Every X is Y	$X \subseteq Y$
A^{I}XY	x)y	Every x is y	$X' \subseteq Y'$
O_{I}XY	X:Y	Some X is not Y	$X \cap Y' \neq \emptyset$
O^{I}XY	x:y	Some x is not y	$X' \cap Y \neq \emptyset$
E_{I}XY	X.Y	No X is Y	$X \cap Y = \emptyset$
E^{I}XY	x.y	No x is y	$X' \cap Y' = \emptyset$
I_{I}XY	XY	Some X is Y	$X \cap Y \neq \emptyset$
I^{I}XY	xy	Some x is y	$X' \cap Y' \neq \emptyset$

Fig. **13** Representative Simple Propositions

By using representative forms the number of simple syllogisms to consider is reduced from 32,768 to $8^3 = 512$, and the number of those that are valid is only 48. This is a very manageable number.

Note that in the expressions in the third column for the propositions F_{I}XY the SYMBOLS X and Y appear; and for the propositions F^{I}XY the contrary SYMBOLS x and y appear. De Morgan prefers to write just A_{I} instead of our A_{I}XY, etc. This makes for compact tables, but the reader has to keep track of the SYMBOLS being used with the form. Our more detailed version should make it clear that A_{I}yZ means 'All y is Z', that A^{I}yZ means 'All Y is z', etc.

The four simple propositions with the lower strokes are the usual categorical propositions of Aristotelian logic (with reference to XY). Actually De Morgan presents his eight simple propositions before giving the equivalences among the 32 original forms. After noting that 'No X is Y' and 'Some X is Y' are convertible De Morgan omits one member from each equivalence class when he presents the equivalences, using his abbreviations above, in the following table (page 61), and says the reader should make a careful study of it. Here he uses the symbol '=' to express 'has the same meaning as', so it is just semantic equivalence:

A_{I}	X)Y	= X.y	= y)x		A^{I}	x)y	=	x.Y	= Y)X
O_{I}	X : Y	= Xy	= y : x		O^{I}	x : y	=	x : Y	= Y : X
E_{I}	X.Y	= X)y	= Y)x		E^{I}	x.y	=	x)Y	= y)X
I_{I}	XY	= X : y	= Y : x		I^{I}	xy	=	x : Y	= y : X

Fig. **14** De Morgan's Table of Equivalent Simple Propositions

To aid the reader in understanding these relationships De Morgan makes use of **graphic** aids like the following for A_{I} (page 61):

$\mathrm{A}_{\mid}$	U	U	U	U	U	U	U	U	U	U	U	U
	X	X	X	X	X	x	x	x	x	x	x	x
	Y	Y	Y	Y	Y	Y	Y	Y	y	y	y	y

Fig. **15** First Visual Aid

This says that there are 12 things in the universe U (the number of columns labelled U), and, of those, 5 are in X and 8 are in Y. Here we see that X)Y is true (every occurrence of an X corresponds to an occurrence of a Y), and furthermore so are X.y and y)x.

On page 63 De Morgan summarizes the relations between the eight forms. This seems to be his version of the Square of Opposition—perhaps we should call it the Table of Opposition for the simple propositions. Only the forms are given, the SYMBOLS being assumed the same for all:

	Denies	Contains	Is indifferent to		Denies	Is contained in	Is indifferent to
$\mathrm{A}_{\mid}$	$\mathrm{O}_{\mid}\mathrm{E}_{\mid}\mathrm{E}^{\mid}$	$\mathrm{I}_{\mid}\mathrm{I}^{\mid}$	$\mathrm{A}^{\mid}\mathrm{O}^{\mid}$	$\mathrm{O}_{\mid}$	$\mathrm{A}_{\mid}$	$\mathrm{E}_{\mid}\mathrm{E}^{\mid}$	$\mathrm{A}^{\mid}\mathrm{O}^{\mid}\mathrm{I}_{\mid}\mathrm{I}^{\mid}$
$\mathrm{A}^{\mid}$	$\mathrm{O}^{\mid}\mathrm{E}^{\mid}\mathrm{E}_{\mid}$	$\mathrm{I}^{\mid}\mathrm{I}_{\mid}$	$\mathrm{A}_{\mid}\mathrm{O}_{\mid}$	$\mathrm{O}^{\mid}$	$\mathrm{A}^{\mid}$	$\mathrm{E}^{\mid}\mathrm{E}_{\mid}$	$\mathrm{A}_{\mid}\mathrm{O}_{\mid}\mathrm{I}^{\mid}\mathrm{I}_{\mid}$
$\mathrm{E}_{\mid}$	$\mathrm{I}_{\mid}\mathrm{A}_{\mid}\mathrm{A}^{\mid}$	$\mathrm{O}_{\mid}\mathrm{O}^{\mid}$	$\mathrm{E}^{\mid}\mathrm{I}^{\mid}$	$\mathrm{I}_{\mid}$	$\mathrm{E}_{\mid}$	$\mathrm{A}_{\mid}\mathrm{A}^{\mid}$	$\mathrm{E}^{\mid}\mathrm{I}^{\mid}\mathrm{O}_{\mid}\mathrm{O}^{\mid}$
$\mathrm{E}^{\mid}$	$\mathrm{I}^{\mid}\mathrm{A}^{\mid}\mathrm{A}_{\mid}$	$\mathrm{O}^{\mid}\mathrm{O}_{\mid}$	$\mathrm{E}_{\mid}\mathrm{I}_{\mid}$	$\mathrm{I}^{\mid}$	$\mathrm{E}^{\mid}$	$\mathrm{A}^{\mid}\mathrm{A}_{\mid}$	$\mathrm{E}_{\mid}\mathrm{I}_{\mid}\mathrm{O}^{\mid}\mathrm{O}_{\mid}$

Fig. **16** De Morgan's Table of Opposition for Simple Propositions

Thus, looking at the first line, we see that $\mathrm{A}_{\mid}$XY contradicts any one of $\mathrm{O}_{\mid}$XY, $\mathrm{E}_{\mid}$XY, and $\mathrm{E}^{\mid}$XY; it implies both $\mathrm{I}_{\mid}$XY and $\mathrm{I}^{\mid}$XY; and it neither implies nor contradicts each of $\mathrm{A}^{\mid}$XY and $\mathrm{O}^{\mid}$XY. Also one sees the symmetry that the introduction of contrary names gives: by shifting the strokes (upper to lower, and vice-versa) one transposes the first two lines of the table as well as the last two lines; and by permuting O with I and A with E one transposes the first and third as well as the second and fourth lines of the table.

1.5. Transformations of Simple Propositions. To give more insight into equivalent propositions De Morgan notes (pages 63–64) that there are certain natural ways to change a proposition into another proposition:[15]

Transformation	Means
S	change the subject to its contrary
P	change the predicate to its contrary
T	transpose the SYMBOLS
F	switch positive and negative
L	do nothing

Fig. **17** De Morgan's Initial Transformations

These transformations, and their compositions, are permutations of the 32 propositions (based on XY). They generate a group of 16 permutations. Essentially De Morgan says the compositions

$$\mathbf{F}, \mathbf{L}, \mathbf{P}, \mathbf{S}, \mathbf{T}, \mathbf{FT}, \mathbf{PF}, \mathbf{PFT}, \mathbf{PT}, \mathbf{SF}, \mathbf{SFT}, \mathbf{SP}, \mathbf{SPF}, \mathbf{SPT}, \mathbf{SPFT}, \mathbf{ST}$$

[15]De Morgan actually uses the letters F, L, P, S, T. This overlaps with his use of capital roman letters for terms, and with our use F, . . . , L. So we have taken the liberty of changing these transformations to bold type.

give the 16 permutations in this group. After observing the generators are of order two he introduces the 16 compositions above in the following equations (page 64):

$$\mathbf{P} = \mathbf{F},\ \mathbf{SP} = \mathbf{SF},\ \mathbf{PF} = \mathbf{L},\ \mathbf{SPF} = \mathbf{S}$$
$$\mathbf{ST} = \mathbf{FT},\ \mathbf{SPT} = \mathbf{FPT},\ \mathbf{SFT} = \mathbf{T},\ \mathbf{SPFT} = \mathbf{PT}$$

This is not a presentation of the group, but rather an assertion that the equated pairs of permutations yield equivalent results when applied to a proposition. Thus, for example, from the second equation one has $(\mathrm{A}_{\mathsf{I}}\mathrm{XY})\mathbf{SP}$ is equivalent to $(\mathrm{A}_{\mathsf{I}}\mathrm{XY})\mathbf{SF}$, that is, $\mathrm{A}_{\mathsf{I}}\mathrm{xy}$ is equivalent to $\mathrm{E}_{\mathsf{I}}\mathrm{xY}$.

These results are combined with others (described below) into two tables (pages 64–65):

L	**T**	**SP**	**SPT**	**L**
PF	**SFT**	**SF**	**PFT**	**PF**

P	**PT**	**S**	**ST**	**P**
F	**SPFT**	**SPF**	**FT**	**F**

One recognizes the original equations as the columns of these tables (with two columns repeated; and he has replaced **FPT** by **PFT**). The second table can be obtained from the first by multiplying through by **P**. Any two (row) adjacent compositions separated by a double line yield equivalent propositions when applied to a *convertible* proposition; and those separated by single lines yields equivalent propositions when applied to an *inconvertible* proposition. Thus, for example, in the second table, **SPF** and **FT** are adjacent and separated by a double line, so applying them to the convertible proposition $\mathrm{E}^{\mathsf{I}}\mathrm{XY}$ gives equivalent results, namely $\mathrm{A}^{\mathsf{I}}\mathrm{xy}$ and $\mathrm{A}^{\mathsf{I}}\mathrm{YX}$.

After some discussion about these tables he has the following rather puzzling passage (page 65):

> It appears, then, that any change which can be made on a proposition, amounts in effect to **L**, **P**, **S**, or **PS**. This is another verification of the preceding table: for all our forms may be derived from applying those which relate to XY in the cases of Xy, xY, and xy.

Unfortunately he does not follow through and apply these permutations to the study of syllogisms. They give (1) a simple way to determine if a particular syllogism is valid, and (2) a routine and fast method of generating a complete catalog of valid syllogisms. Furthermore they are easily extended to apply to the complex propositions that De Morgan introduces. (See Appendix 0 for details.)

1.6. Complex Propositions. Before pursuing simple syllogisms De Morgan introduces seven **complex forms**:

$$\mathrm{C} \qquad \mathrm{C}_{\mathsf{I}} \qquad \mathrm{C}^{\mathsf{I}} \qquad \mathrm{D} \qquad \mathrm{D}_{\mathsf{I}} \qquad \mathrm{D}^{\mathsf{I}} \qquad \mathrm{P}\,.$$

The **complex propositions** (with respect to XY) are as follows (he uses + for **coexists with**, which means 'and'):

Proposition	Definition	Modern Symbolic Rendering
CXY	$\mathrm{E}_{\mathsf{I}}\mathrm{XY} + \mathrm{E}^{\mathsf{I}}\mathrm{XY}$	$\mathrm{X} = \mathrm{Y}'$
$\mathrm{C}_{\mathsf{I}}\mathrm{XY}$	$\mathrm{E}_{\mathsf{I}}\mathrm{XY} + \mathrm{I}^{\mathsf{I}}\mathrm{XY}$	$\mathrm{X} \subset \mathrm{Y}'$
$\mathrm{C}^{\mathsf{I}}\mathrm{XY}$	$\mathrm{E}^{\mathsf{I}}\mathrm{XY} + \mathrm{I}_{\mathsf{I}}\mathrm{XY}$	$\mathrm{X}' \subset \mathrm{Y}$
DXY	$\mathrm{A}_{\mathsf{I}}\mathrm{XY} + \mathrm{A}^{\mathsf{I}}\mathrm{XY}$	$\mathrm{X} = \mathrm{Y}$
$\mathrm{D}_{\mathsf{I}}\mathrm{XY}$	$\mathrm{A}_{\mathsf{I}}\mathrm{XY} + \mathrm{O}^{\mathsf{I}}\mathrm{XY}$	$\mathrm{X} \subset \mathrm{Y}$
$\mathrm{D}^{\mathsf{I}}\mathrm{XY}$	$\mathrm{A}^{\mathsf{I}}\mathrm{XY} + \mathrm{O}_{\mathsf{I}}\mathrm{XY}$	$\mathrm{X}' \subset \mathrm{Y}'$
PXY	$\mathrm{I}_{\mathsf{I}}\mathrm{XY} + \mathrm{I}^{\mathsf{I}}\mathrm{XY} + \mathrm{O}_{\mathsf{I}}\mathrm{XY} + \mathrm{O}^{\mathsf{I}}\mathrm{XY}$	$\mathrm{X} \cap \mathrm{Y} \neq \emptyset \ \wedge\ \mathrm{X} \cap \mathrm{y} \neq \emptyset$ $\wedge\ \mathrm{x} \cap \mathrm{Y} \neq \emptyset \ \wedge\ \mathrm{x} \cap \mathrm{y} \neq \emptyset$

Fig. **18** Complex Propositions

De Morgan simply writes $\mathrm{C} = \mathrm{E}_{\mathsf{I}} + \mathrm{E}^{\mathsf{I}}$, etc. Any pair of these complex propositions contradict each other, so their 'table of opposition' is trivial. For FXY a complex proposition and GXY a simple proposition either GXY follows from FXY, or GXY contradicts FXY.[16] And these seven propositions are, up to equivalence, the only conjunctions of simple propositions with this property.[17]

De Morgan's reasons for introducing complex propositions are: (1) complex propositions are what one uses in everyday speech, for example, when one says 'Some of the responsibility is mine' one means some, but not all; (2) valid complex syllogisms are easier to classify than the valid simple syllogisms; and (3) one can use the classification of the valid complex syllogisms to determine the valid simple syllogisms.

De Morgan notes that every complex proposition is a *conjunction* of simple propositions (as in the table above), and every simple proposition is a *disjunction* of complex propositions. So he says simple and complex propositions are on an equal footing. De Morgan does not give a complete listing of the disjunctive expressions for the simple propositions—we provide one here (using De Morgan's abbreviated notation; he does not introduce a symbol for 'or'):

$A_{\vert}$	=	D or $D_{\vert}$	$I_{\vert}$	=	D or $D_{\vert}$ or $D^{\vert}$ or $C^{\vert}$ or P
$A^{\vert}$	=	D or $D^{\vert}$	$I^{\vert}$	=	D or $D^{\vert}$ or $D_{\vert}$ or $C_{\vert}$ or P
$E_{\vert}$	=	C or $C_{\vert}$	$O_{\vert}$	=	C or $C_{\vert}$ or $C^{\vert}$ or $D^{\vert}$ or P
$E^{\vert}$	=	C or $C^{\vert}$	$O_{\vert}$	=	C or $C_{\vert}$ or $C^{\vert}$ or $D_{\vert}$ or P

Fig. **19** Simple Propositions as Disjunctions of Complex Propositions

De Morgan extends his classification of equivalent propositions to include the complex ones. On page 70 he gives all but the last line of the following master table of equivalent propositions:

XY	YX	xY	Yx	Xy	yX	xy	yx
$A_{\vert}O^{\vert}D_{\vert}$	$A^{\vert}O_{\vert}D^{\vert}$	$E^{\vert}I_{\vert}C^{\vert}$	$E^{\vert}I_{\vert}C^{\vert}$	$E_{\vert}I^{\vert}C_{\vert}$	$E_{\vert}I^{\vert}C_{\vert}$	$A^{\vert}O_{\vert}D^{\vert}$	$A_{\vert}O^{\vert}D_{\vert}$
$A^{\vert}O_{\vert}D^{\vert}$	$A_{\vert}O^{\vert}D_{\vert}$	$E_{\vert}I^{\vert}C_{\vert}$	$E_{\vert}I^{\vert}C_{\vert}$	$E^{\vert}I_{\vert}C^{\vert}$	$E^{\vert}I_{\vert}C^{\vert}$	$A_{\vert}O^{\vert}D_{\vert}$	$A^{\vert}O_{\vert}D^{\vert}$
$E_{\vert}I^{\vert}C_{\vert}$	$E_{\vert}I^{\vert}C_{\vert}$	$A^{\vert}O_{\vert}D^{\vert}$	$A_{\vert}O^{\vert}D_{\vert}$	$A_{\vert}O^{\vert}D_{\vert}$	$A^{\vert}O_{\vert}D^{\vert}$	$E^{\vert}I_{\vert}C^{\vert}$	$E^{\vert}I_{\vert}C^{\vert}$
$E^{\vert}I_{\vert}C^{\vert}$	$E^{\vert}I_{\vert}C^{\vert}$	$A_{\vert}O^{\vert}D_{\vert}$	$A^{\vert}O_{\vert}D^{\vert}$	$A^{\vert}O_{\vert}D^{\vert}$	$A_{\vert}O^{\vert}D_{\vert}$	$E_{\vert}I^{\vert}C_{\vert}$	$E_{\vert}I^{\vert}C_{\vert}$
DCP	DCP	CDP	CDP	CDP	CDP	DCP	DCP

Fig. **20** Master Table of Equivalent Propositions

The difference between the first and second rows, as well as the third and fourth rows, is just a shift of the strokes. An equivalence class of a proposition can be found by choosing a row and then selecting the ith entry from each of the eight columns, where i can be one of 1, 2, or 3. Thus looking at the second row, with $i = 3$, we have the equivalence class

$$D^{|}XY,\ D_{|}YX,\ C_{|}xY,\ C_{|}Yx,\ C^{|}Xy,\ C^{|}yX,\ D_{|}xy,\ D^{|}yx$$

Finally De Morgan gives a table of immediate inferences and denials between simple and complex propositions.

[16]De Morgan actually defines complex propositions as follows (page 65):

> A *complex proposition* is one which involves within itself the assertion or denial of each and all of the eight simple propositions.

[17]In modern terminology, the seven complex propositions KXY are the atoms of the Boolean algebra generated by the eight simple propositions SXY. Since the eight simple propositions SXY are (up to equivalence) closed under negation, the atoms can be expressed as meets of the generators.

1.7. Syllogisms. Let us say that a (simple or complex) proposition φ, using a form F, **is based on** the SYMBOLS XY if φ is any of the eight propositions FXY, FYX, ... , Fyx. A **syllogism**[18] is an argument in one of the two forms

$$\varphi_1, \varphi_2 \begin{bmatrix} \text{imply} \\ \text{deny} \end{bmatrix} \varphi_3$$

The three propositions φ_i must be based on three SYMBOLS, no two of the φ_i being based on the same pair. When no SYMBOLS are mentioned then the default pairs are XY, YZ, XZ. De Morgan abbreviates the **affirmatory** case to $\varphi_1\varphi_2\varphi_3$, but unfortunately he no longer uses a notation for the negation of a proposition, and introduces no notation for a **negatory** syllogism.[19] Sometimes we will write $\varphi_1, \varphi_2, \varphi_3$ in the affirmatory case for clarity.

The syllogism is **simple** if all three φ_i are simple propositions, and it is **complex** if all three φ_i are complex propositions. Otherwise the syllogism is said to be **mixed**.[20] As simple propositions are closed (up to equivalence) under negation, De Morgan only considers the affirmatory simple syllogisms. And for complex syllogisms, if $\varphi_1\varphi_2\varphi_3$ is valid then the premises $\varphi_1\varphi_2$ deny ψ for ψ any complex proposition based on XZ that is not equivalent to φ_3. In this case De Morgan only writes down the affirmatory case $\varphi_1\varphi_2\varphi_3$, and not the *accompanying negatory cases* in his listing of the valid complex syllogisms.

Note that complex propositions have the property that each one based on XY is equivalent to a unique complex FXY. Again we take the propositions FXY to be representative of their equivalence classes.

To find the representative GXY of a given proposition, say Fθ, look in the first column of the master table of equivalent propositions to find F, look at the headers of the columns to find θ, and then find the entry G in the table corresponding to the row of F and the ith entry of the column of θ, where F is the ith entry in the first column. Thus the representative of D^{I}Yx is C_{I}XY as C_{I} is the 3rd entry of the 2nd row and 4th column.

By using the master table of equivalent propositions any syllogism can be readily transformed into a **standard form** syllogism FXY,GYZ,HXZ. De Morgan shortens this to FGH(XYZ), and even to just FGH. The transformation replaces each of the three propositions by its representative, and thus the transformed syllogism is valid iff the original syllogism is valid.

For example, to find the standard form of the syllogism $\mathrm{O}_{\mathsf{I}}\mathrm{yX}, \mathrm{C}^{\mathsf{I}}\mathrm{Zy}, \mathrm{I}_{\mathsf{I}}\mathrm{zx}$ simply find the representatives. The representative of O_{I}yX is, from the table, I_{I}XY. Of course we need one more round of translation for the second and third propositions since they are based on YZ and XZ instead of XY. For the second proposition, translate C^{I}Zy into C^{I}Yx, find its representative D^{I}XY, and then the representative of C^{I}Zy is D^{I}YZ. Doing the same for the third proposition gives the standard form $\mathrm{I}_{\mathsf{I}}\mathrm{D}^{\mathsf{I}}\mathrm{I}^{\mathsf{I}}$.

By using standard forms the total number of syllogisms to consider is $15^3 = 3,475$. De Morgan's plan is to first determine the valid complex syllogisms, then the valid simple syllogisms, and finally the valid mixed syllogisms. To analyze complex syllogisms De Morgan uses a graphical aid described in §1.8. Then to determine the other syllogisms he makes a clever use of the interplay between simple and complex syllogisms.

Before going into the technical details we want to mention that in the chapter On the Syllogism, page 114, De Morgan challenges the adequacy of syllogistic reasoning. He says from 'man is an animal' one can infer 'the head of a man is the head of an animal'. He does not prove this cannot

[18]When De Morgan says 'syllogism' he means 'valid syllogism'—we are not following this convention.

[19]Although he uses (F) in the first chapter for negation, in this chapter parentheses are just used as delimiters, so (F) means the same as F.

[20]Actually he requires one of the premises of a mixed syllogism to be complex, the other simple. He does not discuss the possibility that just the conclusion be simple, as in $\mathrm{D}_{\mathsf{I}}\mathrm{D}^{\mathsf{I}}\mathrm{I}_{\mathsf{I}}$.

be achieved by syllogistic reasoning, and simply offers further challenges to anyone who thinks they can. The solution he offers is to extend Aristotle's dictum de omni et nullo to the following:

> For every term used universally *less* may be substituted, and for every term used particularly, *more*. The species may take the place of the genus, when all the genus is spoken of: the genus may take the place of the species when some of the species is mentioned, or the genus, used particularly, may take the place of the species used universally. Not only in syllogisms, but in all the ramifications of the description of a complex term. Thus for "men who are not Europeans" may be substituted "animals who are not English."

1.8. Complex Syllogisms. So let us start with a complete list of the 48 valid complex syllogisms[21] (in standard form) in a table with the first premise in the left column, the second premise along the top row, and the conclusion being the corresponding entry in the table. A '•' means no conclusion is possible:

	C	D	P	C_{I}	C^{I}	D_{I}	D^{I}
C	D	C	P	D^{I}	D_{I}	C^{I}	C_{I}
D	C	D	P	C_{I}	C^{I}	D_{I}	D^{I}
P	P	P	•	•	•	•	•
C_{I}	D_{I}	C_{I}	•	$:C^{\mathrm{I}}$	D_{I}	$:D^{\mathrm{I}}$	C_{I}
C^{I}	D^{I}	C^{I}	•	D^{I}	$:C_{\mathrm{I}}$	C^{I}	$:D_{\mathrm{I}}$
D_{I}	C_{I}	D_{I}	•	C_{I}	$:D^{\mathrm{I}}$	D_{I}	$:C^{\mathrm{I}}$
D^{I}	C^{I}	D^{I}	•	$:D_{\mathrm{I}}$	C^{I}	$:C_{\mathrm{I}}$	D^{I}

Fig. **21** The 48 Valid Complex Syllogisms

We use De Morgan's notation of $:\mathrm{F}$ to indicate that one has two negatory conclusions that include the 'no stroke' version of F. For example $D_{\mathrm{I}}D^{\mathrm{I}}{:}C^{\mathrm{I}}$ means that the two negatory syllogisms 'D_{I} and D^{I} deny C^{I}' and 'D_{I} and D^{I} deny C' both hold.

De Morgan focuses on deriving the results of the lower right quadrant, using diagrams like the following:

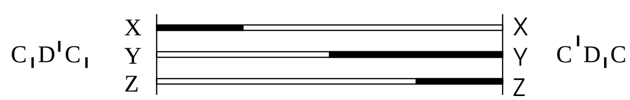


Fig. **22** De Morgan's Second Visual Aid

Each such diagram gives two syllogisms. The dark line segments indicate the portion of the universe that belongs to each term on the left, and the white segments apply to the terms on the right. The syllogism on the left says:

X is properly disjoint from Y
Y is a proper superset of Z
―――――――――――――
X is properly disjoint from Z

He gives four such diagrams, to cover the complex syllogisms that do not mention C, D or P. The syllogisms involving C or D, but not P, are briefly described as limiting cases of these. A separate discussion takes care of the cases that involve P.

[21]Negatory syllogisms that accompany an affirmatory syllogism will not be mentioned, following De Morgan's convention.

1.9. Simple Syllogisms. Using the valid complex syllogisms De Morgan now proceeds to analyze the valid simple syllogisms. First he claims that if one has a particular premise[22] then the conclusion must be particular, and furthermore two particular premises give no conclusion (page 86):

> The following theorems will be necessary;—I. *A particular premise cannot be followed by a universal conclusion.*

This result holds for mixed syllogisms as well. His proof is by a single example to show the method. The idea is as follows. Suppose FGH is a valid syllogism with G particular and H universal. (A similar proof works when F is particular and G universal.) Let KXY be a complex proposition that strengthens FXY, where K is not C, D, or P. Then the syllogism KPH is valid as P strengthens G. As H is universal the proposition HXZ is equivalent to a disjunction of two complex propositions, say 'H_1XZ or H_2XZ', where H_1 is either C or D. But then H_1XZ contradicts the premises KXY,PYZ, so it follows that KPH_2 is valid. But then De Morgan refers to a previous claim (page 85) that says for K one of $\mathrm{C}_\shortmid, \mathrm{C}^\shortmid, \mathrm{D}_\shortmid, \mathrm{D}^\shortmid$ the premises KXY,PYZ are consistent with three different complex propositions MXZ. This gives a contradiction.

> 2. *From two particular premises no conclusion can follow.*

De Morgan's proof is again by a single example to illustrate the method. A general version would be as follows. Suppose FGH is a valid syllogism with F and G particular statements. Then PPH is also valid. Now any simple H will contradict some complex conclusion, but the premises PP are consistent with all complex conclusions. Again a contradiction.

Now he is ready to determine the valid simple syllogisms. Given a valid simple syllogism FGH one can strengthen the premises to complex propositions using only the four forms $\mathrm{C}_\shortmid$, $\mathrm{C}^\shortmid$, $\mathrm{D}_\shortmid$, and $\mathrm{D}^\shortmid$. Suppose we have done this, yielding the valid syllogism KLH. If there is a complex form M such that KLM is a valid syllogism then M is not C, D, or P; and H must be a consequence of M.

There are exactly eight valid affirmatory complex syllogisms not involving C, D, or P, namely:

$$\mathrm{C}_\shortmid\mathrm{C}^\shortmid\mathrm{D}_\shortmid \quad \mathrm{C}_\shortmid\mathrm{D}^\shortmid\mathrm{C}_\shortmid \quad \mathrm{C}^\shortmid\mathrm{C}_\shortmid\mathrm{D}^\shortmid \quad \mathrm{C}^\shortmid\mathrm{D}_\shortmid\mathrm{C}^\shortmid \quad \mathrm{D}_\shortmid\mathrm{C}_\shortmid\mathrm{C}_\shortmid \quad \mathrm{D}_\shortmid\mathrm{D}_\shortmid\mathrm{D}_\shortmid \quad \mathrm{D}^\shortmid\mathrm{C}^\shortmid\mathrm{C}^\shortmid \quad \mathrm{D}^\shortmid\mathrm{D}^\shortmid\mathrm{D}^\shortmid$$

De Morgan observes a fascinating pattern, namely that if one takes any of the above eight triples of forms, say KLM, then by putting the eight syllogisms

$$\mathrm{KLM(XYZ)} \qquad \mathrm{KLM(XYz)} \qquad \ldots \qquad \mathrm{KLM(xyz)}$$

into standard form one obtains the previous list of eight valid complex syllogisms. For our purposes it suffices to check this fact for $\mathrm{D}_\shortmid\mathrm{D}_\shortmid\mathrm{D}_\shortmid$.

De Morgan determines 3 simple syllogisms from $\mathrm{D}_\shortmid\mathrm{D}_\shortmid\mathrm{D}_\shortmid$, and then uses the last observation to find a total of 24 syllogisms. He claims the other syllogisms can be obtained by weakening the conclusions or strengthening the premises of the 24 syllogisms. This claim is correct, but the proof is really left to the reader. We will give an alternate approach to De Morgan's syllogisms in Appendix 0 that makes it fairly easy to verify this claim.

Now the four simple propositions FXY that follow from $\mathrm{D}_\shortmid$XY are:

$$\mathrm{A}_\shortmid\mathrm{XY} \qquad \mathrm{I}_\shortmid\mathrm{XY} \qquad \mathrm{I}^\shortmid\mathrm{XY} \qquad \mathrm{O}^\shortmid\mathrm{XY}$$

De Morgan only uses the first and last of the four—they appear in his definition $\mathrm{D}_\shortmid = \mathrm{A}_\shortmid + \mathrm{O}^\shortmid$—and he gives the three simple syllogisms obtained[23] from $\mathrm{D}_\shortmid\mathrm{D}_\shortmid\mathrm{D}_\shortmid$ (using the two simple propositions just mentioned):

$$\mathrm{A}_\shortmid\mathrm{A}_\shortmid\mathrm{A}_\shortmid \qquad \mathrm{A}_\shortmid\mathrm{O}^\shortmid\mathrm{O}^\shortmid \qquad \mathrm{O}^\shortmid\mathrm{A}_\shortmid\mathrm{O}^\shortmid$$

---

[22] Recall that particular and universal propositions are simple propositions.

[23] Had he used all four propositions he would have found seven simple syllogisms, namely

$$\mathrm{A}_\shortmid\mathrm{A}_\shortmid\mathrm{A}_\shortmid \quad \mathrm{A}_\shortmid\mathrm{A}_\shortmid\mathrm{I}_\shortmid \quad \mathrm{A}_\shortmid\mathrm{A}_\shortmid\mathrm{I}^\shortmid \quad \mathrm{A}_\shortmid\mathrm{I}^\shortmid\mathrm{I}^\shortmid \quad \mathrm{A}_\shortmid\mathrm{O}^\shortmid\mathrm{O}^\shortmid \quad \mathrm{I}_\shortmid\mathrm{A}_\shortmid\mathrm{I}_\shortmid \quad \mathrm{O}^\shortmid\mathrm{A}_\shortmid\mathrm{O}^\shortmid\ .$$

Now applying these three triples of simple forms to the eight triples XYZ, . . . , xyz of SYMBOLS, and putting them in standard form, he obtains the 24 syllogisms in bold type in the following table of the 48 valid simple syllogisms (in standard form):[24]

Universal Syllogism	$\mathbf{A_{\|}A_{\|}A_{\|}}$	$\mathbf{A^{\|}A^{\|}A^{\|}}$	$\mathbf{E_{\|}E^{\|}A_{\|}}$	$\mathbf{E^{\|}E_{\|}A^{\|}}$
Weakened Conclusion	$A_{\|}A_{\|}I_{\|}$	$A^{\|}A^{\|}I^{\|}$	$E_{\|}E^{\|}I_{\|}$	$E^{\|}E_{\|}I^{\|}$
Weakened Conclusion	$A_{\|}A_{\|}I^{\|}$	$A^{\|}A^{\|}I_{\|}$	$E_{\|}E^{\|}I^{\|}$	$E^{\|}E_{\|}I_{\|}$
Universal Syllogism	$\mathbf{A_{\|}E_{\|}E_{\|}}$	$\mathbf{A^{\|}E^{\|}E^{\|}}$	$\mathbf{E_{\|}A^{\|}E_{\|}}$	$\mathbf{E^{\|}A_{\|}E^{\|}}$
Weakened Conclusion	$A_{\|}E_{\|}O_{\|}$	$A^{\|}E^{\|}O^{\|}$	$E_{\|}A^{\|}O_{\|}$	$E^{\|}A_{\|}O^{\|}$
Weakened Conclusion	$A_{\|}E_{\|}O^{\|}$	$A^{\|}E^{\|}O_{\|}$	$E_{\|}A^{\|}O^{\|}$	$E^{\|}A_{\|}O_{\|}$
Strengthened Premise	$A_{\|}A^{\|}I^{\|}$	$A^{\|}A_{\|}I_{\|}$		
	$\mathbf{A_{\|}I^{\|}I^{\|}}$	$\mathbf{A^{\|}I_{\|}I_{\|}}$	$\mathbf{I^{\|}A^{\|}I^{\|}}$	$\mathbf{I_{\|}A_{\|}I_{\|}}$
Strengthened Premise	$A_{\|}E^{\|}O^{\|}$	$A^{\|}E_{\|}O_{\|}$	$E^{\|}A^{\|}O_{\|}$	$E_{\|}A_{\|}O^{\|}$
	$\mathbf{A_{\|}O^{\|}O^{\|}}$	$\mathbf{A^{\|}O_{\|}O_{\|}}$	$\mathbf{O_{\|}A^{\|}O_{\|}}$	$\mathbf{O^{\|}A_{\|}O^{\|}}$
	$\mathbf{E_{\|}I_{\|}O^{\|}}$	$\mathbf{E^{\|}I^{\|}O_{\|}}$	$\mathbf{I_{\|}E_{\|}O_{\|}}$	$\mathbf{I^{\|}E^{\|}O^{\|}}$
Strengthened Premise	$E_{\|}E_{\|}I^{\|}$	$E^{\|}E^{\|}I_{\|}$		
	$\mathbf{E_{\|}O_{\|}I^{\|}}$	$\mathbf{E^{\|}O^{\|}I_{\|}}$	$\mathbf{O^{\|}E_{\|}I^{\|}}$	$\mathbf{O_{\|}E^{\|}I_{\|}}$

Fig. **23** The 48 Valid Simple Syllogisms

Classifications of the valid simple syllogisms by De Morgan are given to the left of the table. Each such classification applies to all the syllogisms in that row. Of the possible 64 premises for a standard form simple syllogism the above table shows that 32 are involved in valid syllogisms. The **universal syllogisms** are those that have all three propositions universal. There are eight such syllogisms. (The other syllogisms are called **particular syllogisms**.) One can weaken the conclusion of each of the eight valid universal syllogisms to a particular proposition in two ways. This gives the rows marked **weakened conclusion**. These are omitted by De Morgan, just as syllogisms with weakened conclusions are omitted in the traditional Aristotelian logic. De Morgan says that if the premises are stronger than needed for a conclusion then such a syllogism should also be omitted. This gives the rows marked **strengthened conclusion**. However he tends to include the strengthened syllogisms, for example when finding the total number (32) of pairs of premises involved in valid simple syllogisms.

The 24 syllogisms in bold type are the **fundamental syllogisms**. These are the ones that are neither weakened nor strengthened. The horizontal lines are included in the table to group the weakened or strengthened syllogisms with fundamental syllogisms from which they can be derived.[25]

1.10. Mixed Syllogisms. In a couple of paragraphs De Morgan gives a list of rules that show how to relate a mixed syllogism to one that is not mixed to decide if the mixed syllogism is valid. For the valid affirmatory syllogisms we refer the reader to Appendix 0, where one finds a simple algorithm to determine if any affirmatory syllogism is valid, and a table of all the standard form valid affirmatory syllogisms (with strongest possible conclusions) using De Morgan's 15 propositional forms.

1.11. Further Comments on Syllogisms. De Morgan shows throughout his work on logic a tremendous fascination with presentation and symbolic notation. The traditional **structure of the syllogism** seems quite unnatural so, for example, he changes the 1st Figure AAA syllogism

[24]De Morgan lists those in bold type—see the 3rd column of page 89. Only a few of the others are explicitly mentioned.

[25]It is interesting to note that the fundamental syllogisms are precisely the simple syllogisms that are valid under modern semantics.

'Every Y is Z, Every X is Y, therefore Every X is Z' into 'Every X is Y, Every Y is Z, therefore Every X is Z'.

Also he invents **semantically loaded** words to describe his complex propositions, words which he says will make it clear whether a syllogism is correct or not:

Form	Expression	Form	Expression
C	**contrary**	D	**identical of**
C_{I}	**subcontrary**	D_{I}	**subidentical of**
C^{I}	**supercontrary**	D^{I}	**superidentical of**

Fig. **24** De Morgan's Terminology for Complex Propositions

Thus reading $\mathrm{C}_{\mathrm{I}}\,\mathrm{C}^{\mathrm{I}}\,\mathrm{D}_{\mathrm{I}}$ as 'Subcontrary of supercontrary is subidentical' is to immediately convey the validity of this syllogism.

At the end of the book, in Chapter XIV, On the Verbal Description of the Syllogism, he tries to do the same for the simple propositions, and devises the following:

Form	Expression	Form	Expression
A_{I}	**species**	O_{I}	**non-species**
A^{I}	**genus**	O^{I}	**non-genus**
E_{I}	**external**	I_{I}	**non-external**
E^{I}	**complement**	I^{I}	**non-complement**

Fig. **25** De Morgan's Terminology for Simple Propositions

One must note that his 'complement' means, in modern terminology, 'any superset of the complement'.

1.12. Quantifiers. The quantifier 'all' is perfectly clear as to its meaning, but De Morgan sees numerous possibilities for 'some' (page 58):

> The relation of the universal quantity to the whole quantity of instances in existence is *definite*, being that whole quantity itself. But the particular quantity is wholly *indefinite*: "Some Xs are Ys" gives no clue to the fraction of all the Xs spoken of, nor to the fraction which they make of all the Ys. Common language makes a certain conventional approach to definiteness, which has been thrown away in works of logic. "Some," usually means a rather large small fraction of the whole; a larger fraction would be expressed by "a good many"; and somewhat more than half expressed by "most"; while a still larger proportion would be expressed by "a great majority" or "nearly all". A perfectly *definite particular*, as to quantity, would express how many Xs are in existence, how many Ys, and how many of the Xs are or are not Ys: as in "70 out of the 100 Xs are among the 200 Ys."

In 1846 De Morgan gives two examples of syllogisms using such a quantifier ([**5**], page 9):

Most of the *Y*s are *X*s	Most of the *Y*s are *X*s
Most of the *Y*s are *Z*s	Most of the *Y*s are not *Z*s
Some of the *X*s are *Z*s	Some of the *X*s are not *Z*s

In his efforts to make the syllogism *definite* De Morgan was particularly proud of his idea of the **numerically definite** proposition, an example of which is given in the passage quoted above. He devotes a modest chapter to this form of proposition and the corresponding syllogisms, but this idea was not destined for success among logicians.

1.13. Compound Names. De Morgan introduces **compound names** on page 115 to describe classes naturally composed of others, namely he uses PQ for what we call the intersection of P and Q, and P,Q for their union. He allows any interation of these binary connectives, giving all disjunctive and conjunctive combinations of his SYMBOLS. His purpose for introducing them is only to reduce other syllogisms to categorical form (see §1.14).

He points out (page 116) that:

> The contrary of PQR is p,q,r; that of P,Q,R is pqr; that of PQ,R is (p,q)r : in contraries, conjunction and disjunction change places.

And then, at the bottom of page 118, we finally have the famous **De Morgan Laws**:

> The contrary of PQ is p,q; that of P,Q is pq.

As a matter of fact all of the laws of Boolean algebra that he discusses, except for one instance of the distributive law described below, are of the form 'The contrary of — is —'. The reason for his focus on the contrary of a compound name is that he wants to convince the reader that this also has a compound name—after all, having names for contraries is one of the key ideas introduced by De Morgan.

On page 119 we find the one exception, a form of the **distributive law**, of intersection over union, stated as follows:

> I need hardly have remarked that (P,Q)(R,S) is PR,PS,QR,QS.

In modern notation this is

$$(\mathrm{P} \cup \mathrm{Q}) \cap (\mathrm{R} \cup \mathrm{S}) \;=\; (\mathrm{P} \cap \mathrm{R}) \cup (\mathrm{P} \cap \mathrm{S}) \cup (\mathrm{Q} \cap \mathrm{R}) \cup (\mathrm{Q} \cap \mathrm{S}).$$

The **associative laws** for union and for intersection are implicit in his omission of parentheses. And presumably the commutative and idempotent laws were too obvious to state. Nor are they used. He does not introduce a symbol to express the fact that two (compound) names are the same until he uses '$\|$' in his Syllabus of 1860. And then he makes very little use of this symbol aside from a few exercises in paragraph 134 (page 182) on contraries, for example X$\|$(A,B)C gives x$\|$(ab,c).

After this introduction to his notation he considers another expansion of his categorical forms by allowing the simplest compound names to appear in them, for example, XY)P,Q means that 'Everything that is both X and Y is either P or Q'. But his work on this is only a few pages, just enough to open up the subject. (His requirement that the subject and predicate be nonempty and not the universe would complicate an analysis.)

De Morgan is surprisingly apologetic for the notation for compound names (page 116):

> With respect to this and other cases of notation, repulsive as they may appear, the reader who refuses them is in one of two circumstances. Either he wants to give his assent or dissent to what is said of the form by means of the matter, which is easing the difficulty by avoiding it, and stepping out of logic; or else he desires to have it in a shape in which he may get that most futile of all acquisitions, called a *general idea*, which is truly, to use the contrary adjective term as colloquially, *nothing particular*, a whole without parts.

Regarding the nature of compound names he says (page 117):

> Whatever has the right to the name P, and also to the name Q, has right to the compound name PQ. This is an absolute identity, for by the name PQ we signify nothing but what has right to both names. According X)P + X)Q = X)PQ is not a syllogism, nor even an inference, but only the assertion of our right to use at our pleasure either one of two ways of saying the same thing instead of the other.

Of course we view $(\mathrm{X} \subseteq \mathrm{P})\&(\mathrm{X} \subseteq \mathrm{Q}) \leftrightarrow (\mathrm{X} \subseteq \mathrm{P} \cap \mathrm{Q})$ as a very simple *theorem* about sets, where we base the set theory as usual on *membership* ($\in$). However he appears to be saying that his expression is a *definition* of PQ. Such an implicit definition would not be acceptable today without an explanation as to why a unique something actually satisfies the definition.[26]

[26] In 1880 Peirce uses similar implicit definitions for his foundations of the algebra of logic. He starts with a poset and defines the operations + and × on a pair of elements as the least upper bound and greatest lower bound. However he does not explain why these bounds should exist. Schröder adopts Peirce's approach in 1890, again without explaining why the bounds exist, as the starting point for his Algebra of Logic.

1.14. Other Syllogisms. Let us look at an example of how De Morgan uses compound names to reduce the constructive dilemma to a categorical syllogism (page 123):

> *Example 2.* "If A be B, E is F; and if C be D, E is F; but either A is B or C is D; therefore E is F."
> This can be reduced to
>
> $$\mathsf{P)R + Q)R + S)P,Q = S)R}$$
>
> which is immediately made a common syllogism by changing P)R + Q)R into P,Q)R.

De Morgan is letting P be the proposition 'A is B', etc. S denotes a true proposition. But to use his previous setup we really need classes, not propositions. The simplest solution is to refer to Whately's treatment where he lets P denote the class of instances where 'A is B' is true, etc. Then the above becomes

$$\mathrm{P,Q)R + U)P,Q = U)R}$$

where U is the universe. However this does seem to conflict with his condition that a term in a syllogism cannot be the empty class or the universe.

After presenting his system De Morgan comments on the rigidity of the followers of Aristotle (page 127):

> FROM the time of Aristotle until now, the formal inference has been a matter of study. In the writings of the great philosopher, and in a somewhat scattered manner, are found the materials out of which was constructed the system of syllogism now and always prevalent: and two distinct principles of exclusion appear to be acted on. Perhaps it would be more correct to say that the followers collected two distinct principles of exclusion from the writings of the master, by help of the assumption that everything not used by the teacher was forbidden to the learner. I cannot find that Aristotle either limits his reader in this manner, or that he anywhere implies that he has exhausted all possible modes of syllogizing. But whether these exclusions are to be attributed to the followers alone, or whether those who have more knowledge of his writings than myself can fix them upon the leader, this much is certain, that they were adopted, and have in all time dictated the limits of the syllogism. Of all men, Aristotle is the one of whom his followers have worshipped his defects as well as his excellencies: which is what he himself never did to any man living or dead; indeed, he has been accused of the contrary fault.

2. The Affirmatory Syllogisms of *Formal Logic*

De Morgan's transformation of propositions can be used to give a simple algorithm for the affirmatory syllogisms. Let us define three operations on the simple and complex forms F by the following, where 'equiv.' means 'is equivalent to':

$$\begin{array}{rll} \overline{\mathrm{F}}\mathrm{XY} & \text{equiv.} & \mathrm{Fxy} \\ \mathrm{F}^{-1}\mathrm{XY} & \text{equiv.} & \mathrm{FYX} \\ \mathrm{F}^{*}\mathrm{XY} & \text{equiv.} & \mathrm{FXy} \end{array}$$

In terms of De Morgan's transformations we have: $\overline{\mathrm{F}}\mathrm{XY}$ is equivalent to (FXY)**SP**, $\mathrm{F}^{-1}\mathrm{XY}$ is equivalent to (FXY)**T**, and $\mathrm{F}^{*}\mathrm{XY}$ is equivalent to (FXY)**P**. These operations are given by the following permutations of the 15 forms:

$$\begin{array}{ll} \overline{\mathrm{F}}: & (\mathrm{A}_{\mathsf{I}}\ \mathrm{A}^{\mathsf{I}})(\mathrm{E}_{\mathsf{I}}\ \mathrm{E}^{\mathsf{I}})(\mathrm{I}_{\mathsf{I}}\ \mathrm{I}^{\mathsf{I}})(\mathrm{O}_{\mathsf{I}}\ \mathrm{O}^{\mathsf{I}})(\mathrm{C}_{\mathsf{I}}\ \mathrm{C}^{\mathsf{I}})(\mathrm{D}_{\mathsf{I}}\ \mathrm{D}^{\mathsf{I}}) \\ \mathrm{F}^{-1}: & (\mathrm{A}_{\mathsf{I}}\ \mathrm{A}^{\mathsf{I}})(\mathrm{O}_{\mathsf{I}}\ \mathrm{O}^{\mathsf{I}})(\mathrm{D}_{\mathsf{I}}\ \mathrm{D}^{\mathsf{I}}) \\ \mathrm{F}^{*}: & (\mathrm{A}_{\mathsf{I}}\ \mathrm{E}_{\mathsf{I}})(\mathrm{A}^{\mathsf{I}}\ \mathrm{E}^{\mathsf{I}})(\mathrm{I}_{\mathsf{I}}\ \mathrm{O}_{\mathsf{I}})(\mathrm{I}^{\mathsf{I}}\ \mathrm{O}^{\mathsf{I}})(\mathrm{C}_{\mathsf{I}}\ \mathrm{D}_{\mathsf{I}})(\mathrm{C}^{\mathsf{I}}\ \mathrm{D}^{\mathsf{I}})(\mathrm{C}\ \mathrm{D}) \end{array}$$

Now we give four transformations of syllogisms (in standard form):

$$\begin{array}{llll} (1) & \mathrm{FGH} & \longrightarrow & \mathrm{F}^{*}\overline{\mathrm{G}}\mathrm{H}^{*} \\ (2) & \mathrm{FGH} & \longrightarrow & \mathrm{F}\mathrm{G}^{*}\mathrm{H}^{*} \\ (3) & \mathrm{FGH} & \longrightarrow & \mathrm{G}^{-1}\mathrm{F}^{-1}\mathrm{H}^{-1} \\ (4) & \mathrm{FGH} & \longrightarrow & \overline{\mathrm{F}}\,\overline{\mathrm{G}}\,\overline{\mathrm{H}} \end{array}$$

Note that (1) gives the standard form of FGH(Xyz), (2) the standard form of FGH(XYz), (3) the standard form of FGH(ZYX), and (4) the standard form of FGH(xyz).

The **reduced form** of a syllogism FGH is the result of successively carrying out the following sequence of four steps:

- If F is not among the A,D,I forms then apply transformation (1).
- If G is not among the A,D,I forms then apply transformation (2).
- If G $\prec$ F, where

$$\mathrm{D} \prec \mathrm{P} \prec \mathrm{D}_{|}\,,\ \mathrm{D}^{|} \prec \mathrm{A}_{|}\,,\ \mathrm{A}^{|} \prec \mathrm{I}_{|}\,,\ \mathrm{I}^{|}\ ,$$

  then apply transformation (3).
- If F has an upperstroke apply transformation (4).

The 13 **primary syllogisms** are:

$$\mathbf{DGG} \qquad \text{for G not a C,E, or O form}$$

$$\mathbf{PA_{|}I_{|}} \quad \mathbf{PA^{|}I^{|}} \quad \mathbf{D_{|}A_{|}D_{|}} \quad \mathbf{A_{|}A_{|}A_{|}} \quad \mathbf{A_{|}I^{|}I^{|}}$$

The primary syllogisms are valid.

Let us say that a syllogism KLM is a **specialization** of FGH if the premises of KLM are at least as strong as those of FGH, and the conclusion of KLM is weaker or equal to that of FGH. Thus if KLM is a specialization of a valid syllogism then KLM is valid.

To quickly check that one proposition is stronger than another one can use the following diagram, where F is below G means FXY is stronger than GXY, i.e, GXY follows from FXY:

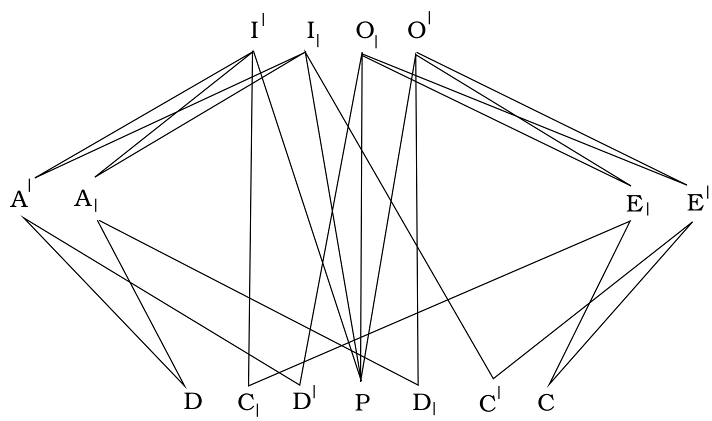


Fig. **26** The 'stronger than' ordering

Theorem A syllogism FGH (in standard form) is valid iff its reduced form is a specialization of one of the primary syllogisms.

Proof The first two steps of the reduction eliminate all occurrences of C,E, and O forms from the premises. The next step ensures *not* (G $\prec$ F). The last step gives an F that does not have an upperstroke. Each of the steps preserves the effects of the previous steps. This leaves the following 27 possibilities for the two premises of a reduced syllogism, where in each row one is to choose the second premise to be any of the bracketed items:

D [D P $\mathrm{D}_{|}$ $\mathrm{D}^{|}$ $\mathrm{A}_{|}$ $\mathrm{A}^{|}$ $\mathrm{I}_{|}$ $\mathrm{I}^{|}$]
P [P $\mathrm{D}_{|}$ $\mathrm{D}^{|}$ $\mathrm{A}_{|}$ $\mathrm{A}^{|}$ $\mathrm{I}_{|}$ $\mathrm{I}^{|}$]
$\mathrm{D}_{|}$ [$\mathrm{D}_{|}$ $\mathrm{D}^{|}$ $\mathrm{A}_{|}$ $\mathrm{A}^{|}$ $\mathrm{I}_{|}$ $\mathrm{I}^{|}$]
$\mathrm{A}_{|}$ [$\mathrm{A}_{|}$ $\mathrm{A}^{|}$ $\mathrm{I}_{|}$ $\mathrm{I}^{|}$]
$\mathrm{I}_{|}$ [$\mathrm{I}_{|}$ $\mathrm{I}^{|}$]

It is trivial to find the best conclusion for the 8 cases in the first row, namely DGG has the strongest conclusion that one can draw from the premises DG. This gives 8 primary syllogisms. The last row yields no valid syllogisms as the premises are both particular. A detailed analysis of the 17 pairs of premises in the second, third and fourth rows will yield the remaining 5 primary syllogisms. Using this it is not difficult to fill in (by hand!) the following table of the 184 standard form valid affirmatory syllogisms that have strongest possible conclusions, using the 15 propositional forms of De Morgan. And from this one can readily determine that there are a total of 424 standard form valid affirmatory syllogisms in De Morgan's system.[27] As usual, the first premise is in the left column, the second premise is in the top row, and the conclusion is the corresponding entry in the table. Several of the syllogisms involving P as a premise have two strongest possible conclusions, and they are listed together. Thus both $D_\shortmid P I^\shortmid$ and $D_\shortmid P O^\shortmid$ are in the collection of valid syllogisms with strongest possible conclusions. The bold type gives the primary syllogisms:

	D	$D_\shortmid$	$D^\shortmid$	C	$C_\shortmid$	$C^\shortmid$	P	$A_\shortmid$	$A^\shortmid$	$E_\shortmid$	$E^\shortmid$	$I_\shortmid$	$I^\shortmid$	$O_\shortmid$	$O^\shortmid$
D	**D**	$\mathbf{D}_\shortmid$	$\mathbf{D}^\shortmid$	C	$C_\shortmid$	$C^\shortmid$	**P**	$\mathbf{A}_\shortmid$	$\mathbf{A}^\shortmid$	$E_\shortmid$	$E^\shortmid$	$\mathbf{I}_\shortmid$	$\mathbf{I}^\shortmid$	$O_\shortmid$	$O^\shortmid$
$D_\shortmid$	$D_\shortmid$	$D_\shortmid$	$I^\shortmid$	$C_\shortmid$	$C_\shortmid$	$O^\shortmid$	$I^\shortmid O^\shortmid$	$\mathbf{D}_\shortmid$	$I^\shortmid$	$C_\shortmid$	$O^\shortmid$	•	$I^\shortmid$	•	$O^\shortmid$
$D^\shortmid$	$D^\shortmid$	$I_\shortmid$	$D^\shortmid$	$C^\shortmid$	$O_\shortmid$	$C^\shortmid$	$I_\shortmid O_\shortmid$	$I_\shortmid$	$D^\shortmid$	$O_\shortmid$	$C^\shortmid$	$I_\shortmid$	•	$O_\shortmid$	•
C	C	$C^\shortmid$	$C_\shortmid$	D	$D^\shortmid$	$D_\shortmid$	P	$E^\shortmid$	$E_\shortmid$	$A^\shortmid$	$A_\shortmid$	$O^\shortmid$	$O_\shortmid$	$I^\shortmid$	$I_\shortmid$
$C_\shortmid$	$C_\shortmid$	$O^\shortmid$	$C_\shortmid$	$D_\shortmid$	$I^\shortmid$	$D_\shortmid$	$I^\shortmid O^\shortmid$	$O^\shortmid$	$C_\shortmid$	$I^\shortmid$	$D_\shortmid$	$O^\shortmid$	•	$I^\shortmid$	•
$C^\shortmid$	$C^\shortmid$	$C^\shortmid$	$O_\shortmid$	$D^\shortmid$	$D^\shortmid$	$I_\shortmid$	$I_\shortmid O_\shortmid$	$C^\shortmid$	$O_\shortmid$	$D^\shortmid$	$I_\shortmid$	•	$O_\shortmid$	•	$I_\shortmid$
P	P	$I_\shortmid O^\shortmid$	$I^\shortmid O_\shortmid$	P	$I^\shortmid O_\shortmid$	$I_\shortmid O^\shortmid$	•	$\mathbf{I}_\shortmid O^\shortmid$	$\mathbf{I}^\shortmid O_\shortmid$	$I^\shortmid O_\shortmid$	$I_\shortmid O^\shortmid$	•	•	•	•
$A_\shortmid$	$A_\shortmid$	$D_\shortmid$	$I^\shortmid$	$E_\shortmid$	$C_\shortmid$	$O^\shortmid$	$I^\shortmid O^\shortmid$	$\mathbf{A}_\shortmid$	$I^\shortmid$	$E_\shortmid$	$O^\shortmid$	•	$\mathbf{I}^\shortmid$	•	$O^\shortmid$
$A^\shortmid$	$A^\shortmid$	$I_\shortmid$	$D^\shortmid$	$E^\shortmid$	$O_\shortmid$	$C^\shortmid$	$I_\shortmid O_\shortmid$	$I_\shortmid$	$A^\shortmid$	$O_\shortmid$	$E^\shortmid$	$I_\shortmid$	•	$O_\shortmid$	•
$E_\shortmid$	$E_\shortmid$	$O^\shortmid$	$C_\shortmid$	$A_\shortmid$	$I^\shortmid$	$D_\shortmid$	$I^\shortmid O^\shortmid$	$O^\shortmid$	$E_\shortmid$	$I^\shortmid$	$A_\shortmid$	$O^\shortmid$	•	$I^\shortmid$	•
$E^\shortmid$	$E^\shortmid$	$C^\shortmid$	$O_\shortmid$	$A^\shortmid$	$D^\shortmid$	$I_\shortmid$	$I_\shortmid O_\shortmid$	$E^\shortmid$	$O_\shortmid$	$A^\shortmid$	$I_\shortmid$	•	$O_\shortmid$	•	$I_\shortmid$
$I_\shortmid$	$I_\shortmid$	$I_\shortmid$	•	$O_\shortmid$	$O_\shortmid$	•	•	•	$I_\shortmid$	•	$O_\shortmid$	•	•	•	•
$I^\shortmid$	$I^\shortmid$	•	$I^\shortmid$	$O^\shortmid$	•	$O^\shortmid$	•	$I^\shortmid$	•	$O^\shortmid$	•	•	•	•	•
$O_\shortmid$	$O_\shortmid$	•	$O_\shortmid$	$I_\shortmid$	•	$I_\shortmid$	•	$O_\shortmid$	•	$I_\shortmid$	•	•	•	•	•
$O^\shortmid$	$O^\shortmid$	$O^\shortmid$	•	$I^\shortmid$	$I^\shortmid$	•	•	•	$O^\shortmid$	•	$I^\shortmid$	•	•	•	•

Fig. **27** Affirmatory Syllogisms with Strongest Conclusions
(Using De Morgan's 15 Propositional Forms)

3. On the Syllogism

Starting in 1846, and continuing until 1863, De Morgan wrote a series of papers called On the Syllogism: I-VI; and an outline paper called Syllabus of a Proposed System of Logic in 1860 that had 244 numbered paragraphs describing his proposed system of logic. These, and an abridged version of an 1860 article Logic for the English Cyclopedia, were collected together in 1966 in a book titled On the Syllogism. (The introduction of this book has an excellent biography of De Morgan.) The page numbers we quote for De Morgan's work from these articles will refer to the page numbering in this book, and not to the original articles. Our references to 'paper I', etc., will refer to his articles 'On the Syllogism, I', etc.

One can summarize the series of six papers as being a continuation of his book; actually the book was an expansion on the first paper, with many of the details we have discussed in the section on Formal Logic originally appearing in a condensed form in this paper—the major exceptions

[27] Each of the 48 complex conclusions can be weakened in 4 ways; each of the 24 universal conclusions can be weakened in 2 ways; and the 112 particular conclusions cannot be weakened.

being the abstract copula and compound names. Also it should be noted that the notation changes considerably from the 1846 paper to the 1847 book.

In paper II (1850), On the Symbols of Logic, the Theory of the Syllogism, and in particular of the Copula, De Morgan improves his symbolism for simple propositions, and introduces general binary relations. Thus in the years 1846 to 1850 we see all of his main ideas for improving logic. First let us discuss his cuneiform-like notation for simple propositions and syllogisms.

3.1. New Notation for Simple Syllogisms. In the 1850 paper De Morgan develops a simple calculus of inference for simple syllogisms based on the following notation:

proposition	notation	proposition	notation
$A_{\vert}XY$	$X))Y$	$I_{\vert}XY$	$X()Y$
$A^{\vert}XY$	$X((Y$	$I^{\vert}XY$	$X)(Y$
$E_{\vert}XY$	$X).(Y$	$O_{\vert}XY$	$X(.(Y$
$E^{\vert}XY$	$X(.)Y$	$O^{\vert}XY$	$X).)Y$

Fig. **28** De Morgan's 1850 Notation for Simple Propositions

To find the propositions equivalent to a given proposition there are two rules:

- Transposing the subject and predicate, and reversing the parentheses, gives an equivalent proposition. For example, $X).)y$ is equivalent to $y(.(X$.
- Changing a SYMBOL to its contrary, reversing the neighboring parenthesis, and adding a dot if there is no dot, otherwise deleting a dot, gives an equivalent proposition. For example $X))Y$ is equivalent to $x(.)Y$, and this in turn is equivalent to $x((y$.

De Morgan also develops a compact notation for syllogisms in standard form. A single example should suffice to explain this. The premises 'No X is Y' and 'No y is z', which in standard form are $E_{\vert}XY$ and $E^{\vert}YZ$, are written in the new notation as $X).(Y$ and $Y(.)Z$. Amalgamate the pair of premises into $X).(Y(.)Z$, and then remove the SYMBOLS X, Y, Z to obtain just $).((.)$. The strongest possible conclusion is $X))Z$, so he abbreviates the syllogism to $).((.) =))$. Note that the conclusion can be obtained by deleting the inner two parentheses and the two dots. This turns out to be part of a general rule. He gives the following two canons to determine if a pair of premises expressed in compact form in the new notation actually has a conclusion, and if so, a deletion algorithm to find the strongest possible conclusion (page 40):

- **(De Morgan's Canon of Validity)** If both premises are universal, or if one is universal and the middle parentheses turn the same way, then there is a conclusion.
- **(De Morgan's Canon of Inference)** If the premises have a conclusion then the strongest conclusion can be found in the following manner: delete the dots if there are two dots, and delete the inner two parentheses.

To see that these canons are correct one only needs to examine the following table that shows all the standard form valid simple syllogisms (with strongest possible conclusions) in this new notation. The first premise is along the left column, the second premise in the top row, and the conclusion, if there is one, in the corresponding entry of the table. A '•' means no conclusion is possible.

		A_{I}	A^{I}	E_{I}	E^{I}	I_{I}	I^{I}	O_{I}	O^{I}
		))	((	).(	(.)	( )	)(	(.(	).)
A_{I}	))	))	)(	).(	).)	•	)(	•	).)
A^{I}	((	( )	((	(.(	(.)	( )	•	(.(	•
E_{I}	).(	).)	).(	)(	))	).)	•	)(	•
E^{I}	(.)	(.)	(.(	((	( )	•	(.(	•	( )
I_{I}	( )	( )	•	(.(	•	•	•	•	•
I^{I}	)(	•	)(	•	).)	•	•	•	•
O_{I}	(.(	•	(.(	•	( )	•	•	•	•
O^{I}	).)	).)	•	)(	•	•	•	•	•

Fig. **29** The Fundamental and Strengthened Simple Syllogisms

De Morgan's new notation was evidently inspired by studying the system of Hamilton based on introducing the quantification of the predicate:

quantifier	subject	copula	quantifier	predicate

(Hamilton did not have names for contraries.) There is a considerable overlap between the eight propositions that Hamilton obtains, and the eight simple propositions of De Morgan. Two of Hamilton's categorical propositions are not among De Morgan's simple propositions: 'All X is all Y' (which is equivalent to De Morgan's complex DXY), and 'Some X is not some Y', which De Morgan finds rather useless.

Hamilton devised a notation for syllogisms that allowed one to immediately read off the conclusion. This system was highly praised and discussed in a publication of William Thomson in 1849. He used a colon ':' for the quantifier 'all', a comma ',' for the quantifier 'some', and a solid wedge to indicate the direction in which to read a proposition. For example the proposition

Fig. **30** Example of Hamilton's Notation for Propositions

is to be read 'Some Y is all X'. A stroke through the wedge is used for the copula 'is not'.

Three examples of Hamilton's syllogisms in this notation are given in De Morgan's paper—two of them are shown here (page 33):

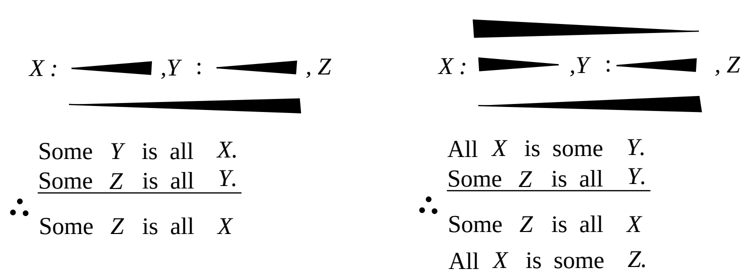


Fig. **31** Examples of Hamilton's Syllogisms

De Morgan notes that Hamilton's notation and algorithm look much like his, but insists they are not so closely related because De Morgan's parentheses do not stand for fixed quantifiers. De Morgan says a term X in a proposition is **universal** if, when showing the validity of the proposition by examining individual elements, one must examine all the elements of X. Otherwise the term is **particular**. For example, to verify the proposition $I^{\mathsf{I}}XY$ one needs to find an element

α not in X and not in Y. To check that α has this property by comparing elements one must examine every element of both X and Y. Thus X and Y both enter universally in this proposition.

This is the sense of quantification that De Morgan says his notation captures:

> Let the subject and predicate, when specified, be written before and after the symbols of quantity. Let the inclosing parenthesis, as in $X)$ or $(X$, denote that the name-symbol X, which would be inclosed if the oval were completed, enters universally. Let an excluding parenthesis as in $)X$ or $X($, signify that the name-symbol enters particularly. Let an even number of dots, or none at all, inserted between the parentheses, denote affirmation or agreement; let an odd number, usually one, denote negation or non-agreement. Thus $X))Y$ means that all Xs are Ys; $X(.(Y$ means that some Xs are not Ys; . . .

It would appear from De Morgan's two examples that one only needs to read $X)$ or $(X$ as 'All X', and $X($ or $)X$ as 'Some X'. But this reading works reasonably well in only six of the eight simple propositions, the ones where Hamilton's propositions are also simple propositions. In the other two cases this reading fails, so one needs to exercise caution: $X)(Y$ is the proposition 'Some x is y', and not Hamilton's 'All X is all Y'; and $X(.)Y$ is the proposition 'No X is Y', and not Hamilton's dubious 'Some X is not some Y'. It is probably better to avoid a direct quantifier reading of the parentheses and rely instead on the table of definitions.

3.2. Developments with Binary Relations. In paper II (1850) De Morgan takes the rather large step of generalizing the copula to an arbitrary **binary relation**, symbolized by either a solid line segment or a dashed line segment, and permitting the copulas in the two premises to be distinct. Almost all of his work using a general binary relation for a copula is for doubly singular propositions, meaning that the subject and predicate are singletons. In this case the quantifiers are not needed since 'All' and 'Some' would have the same meaning. Following De Morgan (page 235) let us say that a syllogism built from doubly singular propositions is a **unit syllogism**. Note that in the study of unit syllogisms one does not want to introduce contraries of names (unless the universe has only two members). So for the study of unit syllogisms all SYMBOLS will be capital letters X, Y, etc.

In order to have the most fundamental unit syllogism 'X is Y, Y is Z, therefore X is Z' hold he needs to introduce the **composition** of the two binary relations as the copula of the conclusion. (He briefly calls this a **bicopular** syllogism.)

He does little with this idea in 1850 besides introduce the composition and contraries of binary relations. It will be ten years before he is prepared to discuss syllogisms using this level of generality. But in 1850 he does argue for the naturalness of using different copulas in a syllogism (page 59):

> The admission of relation in general, and of the composition of relation, tends to throw light upon the difference between the invented syllogism of the logicians and the natural syllogism of the external world. The logician, tied to a verb of identity, from which if he wander it is never quite out of sight, is bound to subject and predicate of the same class; objective both, or subjective both. He cannot say the rose *is* red, for his *is* would require the inference that some red is the rose. He has nothing but a method of reducing his predicate to an object: the rose is a red thing; some red thing is a rose. The common man uses a copula which ties the object up in relation to a more subjective predicate; not reading inversely by intension, not dwelling on redness as an attribute of a rose, but directly by extension, thinking of the family *rose* as his external object, and the sensation red as one condition under which it appears to his senses. Again, an ordinary person says that the rose is red, and red is pretty, so that the rose is pretty. . . .

And, following the last statement, he says that the ordinary person is really dealing with different copulas in such a syllogism.

In 1860, in paper IV On the Logic of Relations, he returns to the use of a binary relation, now symbolized by a letter instead of a line segment, for the copula. For L a binary relation he writes $X..LY$ to signify that 'X **is in the relation** L **to** Y', and $X.LY$ means 'X **is not in the relation**

L **to** Y'. If L is any *transitive* binary relation then De Morgan has the valid unit syllogism:

$$\begin{array}{c} X..LY \\ \underline{Y..LZ} \\ X..LZ \end{array}$$

By using the **compound relation**[28] LM, where $X..LMZ$ means there is a Y such that $X..LY$ and $Y..MZ$, he has the valid unit syllogism:

$$\begin{array}{c} X..LY \\ \underline{Y..MZ} \\ X..LMZ \end{array}$$

To explain some of the notation that De Morgan uses in his theory of syllogisms we give the following definitions (expressed in modern notation):

notation		definition
LY	$=$	$\{\alpha : \{\alpha\}LY\}$
XL	$=$	$\{\beta : XL\{\beta\}\}$

De Morgan uses LY, but not, it seems, XL. Now let us introduce some of his key definitions (pages 220–222), using modern notation when more convenient:

name	notation	definition
converse of L	L^{-1}	$X..L^{-1}Y$ iff $Y..LX$
	LM'	$X..LM'Y$ iff $\emptyset \neq MY \subseteq XL$
	L,M	$X..L,MY$ iff $\emptyset \neq XL \subseteq MY$
contrary of L	l	XlY iff $X.LY$

Fig. **32** De Morgan's Operations on Relations

In a footnote regarding LM' and L,M he says (page 221):

> . . . until this suggestion [to introduce LM' and L,M] arrived, all my efforts to make a scheme of syllogism were wholly unsuccessful.

It is not clear what he means by this—it seems that he has a perfectly satisfactory scheme of unit syllogisms without them, and a quite unsatisfactory discussion of the Aristotelian syllogisms with or without them.

He uses his sign '$\|$' for equality, introduced in his Syllabus, more freely with relations than he does with classes. However, when not directly quoting De Morgan, we prefer to use the usual equality symbol. Also it is easier to summarize his equations between relations if we introduce the notation $\overline{L}$ for the contrary of a relation L (instead of using his l).

On pages 223—224 he describes a number of facts about relations including the following:

- $L \subseteq M$ implies $L^{-1} \subseteq M^{-1}$
- $L \subseteq M$ implies $\overline{M} \subseteq \overline{L}$
- $\overline{\overline{L}} = L$
- $(L^{-1})^{-1} = L$
- $\overline{L^{-1}} = \overline{L}^{-1}$.
- $(LM)^{-1} = M^{-1}L^{-1}$
- $\overline{LM'} = \overline{L},\overline{M}$
- $\overline{LM} = \overline{L}M'$
- $(LM')^{-1} = M^{-1},L^{-1}$

[28] Now called the **relational product** of L and M

From these his other identities and inclusions can be easily deduced.

In this general setting De Morgan gives a table for the 16 unit syllogisms that have the first premise one of 'X is [is not] Y' or 'Y is [is not] X', and the second premise one of 'Y is [is not] Z' or 'Z is [is not] Y'. One can easily put the premises into a standard affirmative form using $Y..LX$ is equivalent to $X..L^{-1}Y$, and $X.LY$ is equivalent to $X..\overline{L}Y$. Then after putting the premises into such a form one has only to apply the basic unit syllogism: from $X..LY$ and $Y..MZ$ follows $X..LMZ$. Let us examine a sample of the 16 cases, expressed in De Morgan's notation (page 232):

$$\begin{array}{l} X..LY \\ Z.MY \\ \hline X..Lm^{-1}Z \\ \\ X.lm^{-1\prime}Z \\ X.L,M^{-1}Z \\ \\ L^{-1}N||M^{-1} \end{array}$$

Fig. **33** An Example of De Morgan's Unit Syllogisms

The premises $X..LY$ and $Z.MY$ are first converted into $X..LY$ and $Y..\overline{M}^{-1}Z$, and the conclusion is then $X..L\overline{M}^{-1}Z$. This explains the third line above.

But then De Morgan wants to cast the conclusion into a negative proposition (for unspecified reasons). If $X..RZ$ is the affirmative conclusion then one has the equivalent negative conclusion $X.\overline{R}Z$. This is where De Morgan uses the operations LM' and L,M to give two different versions of $X.\overline{R}Z$. In the example with $R = L\overline{M}^{-1}$ he expresses $\overline{R}$ as $\overline{L}\,\overline{M}^{-1\prime}$ as well as L,M^{-1}. This gives the 4th and 5th lines of the example above.

Furthermore he wants to add another conclusion involving a relation N that is defined to be the same as R, where $X..RZ$ is the affirmative conclusion, when the premises are both affirmative or both negative; and otherwise N is $\overline{R}$. It is not clear why this should be of interest—perhaps it is a remnant of some earlier attempts to formulate a conclusion. The unexpressed rule seems to be that one should find an equation involving a single product $UV = W$ where one of U, V, W is N; another is either L, $\overline{L}$, L^{-1}, or $\overline{L}^{-1}$, and the third one is one of M, $\overline{M}$, M^{-1}, $\overline{M}^{-1}$. In the example above he has $L^{-1}N = M^{-1}$.

Then he turns to a discussion of how binary relations as copulas fit in with the traditional quantified syllogisms. Before proceeding let us be clear about what De Morgan means when he states a quantified categorical proposition with a general binary relation L as the copula—we will assume the following formulation is correct:

proposition	modern version
All X is Y	$(\forall\alpha \in X)(\exists\beta \in Y)\,(\alpha L\beta)$
No X is Y	$(\forall\alpha \in X)(\forall\beta \in Y)\,(\alpha\overline{L}\beta)$
Some X is Y	$(\exists\alpha \in X)(\exists\beta \in Y)\,(\alpha L\beta)$
Some X is not Y	$(\exists\alpha \in X)(\forall\beta \in Y)\,(\alpha\overline{L}\beta)$

This part of his work, generalizing the quantified syllogisms, is rather disappointing—basically he says they are not so interesting, and that they involve complications that he really does not want to discuss (page 234):

> I enter on this part of the subject only so far as to illustrate the ancient or Aristotelian syllogism. Though of necessity a part of logic, as involving possible forms and necessary connexions, the quantified syllogism of relation is not of primary importance as an explanation of actual thought: for by

> the time that we arrive at the consideration of relation in general we are clear of all necessity for quantification. And for this reason: quantification itself only expresses a relation.

With this dismissal of quantifiers De Morgan misses an essential ingredient of modern logic. He continues:

> Thus if we say that some Xs are connected with Ys, the relation of the class X to the class Y is that of a *partial connexion*: that some at least, all it may be, are connected, is itself a connexion between the *classes*.

This is the first time that De Morgan clearly indicates that he is aware that his propositions $X))Y$, etc., actually *define relations between classes*. Previously he just said that these were *abbreviations*. And he continues:

> Nevertheless, it may be useful to exhibit the modifying quantification as a component, not as inseparably thought of in the compound; though in this we must confine ourselves to what may be called the *Aristotelian* branch of the extended subject.

Two decades later Frege and Peirce would revolutionize logic by properly isolating the 'modifying quantification'. The reason for De Morgan's restriction to the 'Archimedian branch' is that introducing names for contraries creates problems:

> If we would enter completely upon quantified forms, we must examine not only the relation and its contrary, but the relation of a term in connexion with the relation of the contrary term. And here we find that all universal connexion ceases. The repugnance [i.e., disjointness] of X and not-X or x, which, joined with alternance [i.e., the union is U], is the notion the symbols X and x were invented to express, cannot be predicated of LX and Lx: for $Y..LX$ and $Y..Lx$ may coexist. The complete investigation would require subordinate notions of form, effecting the subdivisions of matter.

This quote seems directed at preserving the equivalence of propositions like 'No X is Y' and 'All X is y'. Using arguments similar to those in §1.3 the equivalence of 'No $X..LY$' and 'All $X..Ly$' would hold iff L is a one-one mapping from U to U. Even if one weakens the condition to requiring that 'No $X..LY$' and 'All $X..L^*y$' are equivalent for some choice of copula L^* there are still severe restrictions on L (and L^*).

Thus De Morgan does not even initiate a program to examine the expression of his *simple* syllogisms in the context of bicopular syllogisms. And with the Aristotelian syllogisms he is rather sketchy—he says that the sixteen unit syllogisms he has presented lead to valid quantified syllogisms. He gives a single example, leaving it to the reader to work out the extent to which the Aristotelian inferences are preserved in this general setting. There are severe complications. For example with the 1st Figure EAE syllogism one wants to find a binary relation $\Box$ that makes the following valid:

$$\begin{array}{c}\text{All } X..LY\\ \text{All } Y.MZ\\ \hline \text{All } X.\Box Z\end{array}$$

Of course there is the trivial solution, letting $\Box$ be the empty relation. But there does not seem to be any interesting solution for $\Box$.

After this discussion of quantified bicopular syllogisms De Morgan returns briefly to unit syllogisms, this time with the copula of each of the premises being either '$.L$' or '$..L$', where L is a transitive relation, and in which the conclusion is in one of the four forms 'X .. [.] L [L^{-1}] Y'.

There is a curious discussion of **convertible** relations (i.e., symmetric relations) (page 225):

> And, L being any relation whatever, LL^{-1} is convertible: ... So far as I can see, every convertible relation can be reduced to the form LL^{-1}.

This is clearly false, for consider the convertible relation $\neq$. De Morgan's condition $M = LL^{-1}$ implies that XMX for any X such that for some Y we have XMY. This is essentially noted by De Morgan (page 226):

> Among the subjects of a convertible relation must usually come the predicate itself, unless it be forced out by express convention. If all convertible relation can be expressed by LL^{-1} this is obviously necessary: for $LL^{-1}X$ includes X. Is a man his own brother? It is commonly not so held: but we cannot make a definition which shall by its own power exclude him, unless under a clause expressly framed for the purpose. ... I shall hold, for logical purposes, that the predicate *is* included among its own convertible relatives.

He seems to be saying that the only symmetric relations that he wants to consider are also reflexive.[29]

General binary relations were not an accepted part of logic during his lifetime. In 1860 he says in paper IV ([**5**], page 208):

> Much has been written on relation in all its psychological aspects except the logical one, that is, the analysis of necessary laws of thought connected with the notion of relation. The logician has hitherto carefully excluded from his science the study of relation in general: he places it among those heterogeneous *categories* which turn the porch of his temple into a magazine of raw material mixed with refuse.

The pursuit of the syllogism turned out to be a dead-end of investigation, but the idea of studying general binary relations, and having compound names for relations, would captivate Peirce, who had received a complimentary copy of On the Logic of Relations from De Morgan, and would lead him to develop the powerful calculus of relations (or relatives).

4. Concluding Remarks

De Morgan's description of his system of simple and complex propositions in Formal Logic is given in most appealing terms (pages ix,x):

> A simple notation, which includes the common one, gives the means of representing every syllogism by three letters, each accented above or below. By inspection of one of these symbols it is seen immediately, 1. What syllogism is represented, 2. Whether it be valid or invalid, 3. How it is at once to be written down, 4. What axiom the inference contains, or what is the act of the mind when it makes that inference ...

But his successors would not be so generous in their evaluation. De Morgan's work contributed to the 'air of change' that existed in the 1840's, but his system of logic was regarded as notationally too complex, and, in comparison to Boole's work, hopelessly rooted in the past in focusing on the syllogism, albeit in a more general setting. A number of new ideas were introduced by him, but the interesting ones for the future of logic, like compound names and binary relations, were really not developed in significant detail by De Morgan. His strengths were his ability to ask probing questions, to introduce interesting definitions, and his ability to create compact notation and to compress data into tables.

Jevons summarized the shortcomings of De Morgan's work in 1880 ([**10**], pages xii-xiii):

> After a careful renewed study of the writings of these eminent logicians I felt compelled in the first place to discard the diverse and complicated notative methods of De Morgan ... to import his 'mysterious spiculae' into this book was to add a needless stumbling-block ... There was in fact an unfortunate want of power of generalization in De Morgan; his mind could dissect logical questions into their very atoms, but he could not put the atoms of thought together again into a real system.

Perhaps this is a good place to clarify the limitations of De Morgan's work.

[29] De Morgan's conjecture about the possibility of decomposing a symmetric and reflexive binary relation M into the form LL^{-1} is correct, provided the universe U is infinite. Then one can do this by assigning to each edge E of M a unique element $f(E)$ of U. Then let L be the relation of all pairs $(\alpha, f(E))$ where α is a vertex of E. However on a five-element universe one can find symmetric and reflexive relations M that have no such decomposition. A necessary and sufficient condition for such a decomposition is (thinking of M as a reflexive graph) that one can find no more than $|U|$ cliques in M such that every edge of M belongs to one of these cliques.

- He generalizes the simple propositions by adding certain conjunctions of them called complex propositions to his list of propositions. He says one can express the simple propositions as disjunctions of complex ones, and he briefly has a notation for the denial of a proposition. But he does not take the step of allowing arbitrary Boolean combinations of his propositions to be propositions. This would have yielded a 128 element Boolean algebra of Boolean combinations $\varphi(X, Y)$ of the eight simple propositions, with the complex propositions being the atoms.
- He introduces contraries, union, and intersection, and then constructs compound names. And because he wants to make sure that he has contraries for the compound names the De Morgan laws are mentioned, but there is no deliberate investigation of the laws of compound names. Without a symbol for equality between compound names for most of his work, and without a symbol for complements, he is missing the opportunity to develop one of the vital directions of logic, an equational calculus of compound names.

  De Morgan has Boolean combinations of classes, but not the Boolean algebra of classes. Very little is done towards studying valid syllogisms that use compound names. And he does not notice the parallels between the operations contrary-union-intersection on classes and denial-disjunction-conjunction on propositions.

  It is his student Jevons who, in 1864, starts to fully investigate the laws of (and rules of inference for) compound names. And it is Jevons who shows that this Boolean algebra provides a viable alternative to Boole's algebra of logic.
- He introduces the general binary relation into logic, but he does not see that one can develop a logic for relations that parallels that of the logic for classes, and does not realize that relations will allow one to develop a logic for complicated arguments like those used in mathematics. Rather than a general study of how binary relations can interact, working towards modern logic, De Morgan is content to squeeze them into the traditional syllogistic arguments. The idea of developing a logic for relations rather than for classes would be initiated by C.S. Peirce starting in 1870. (In 1903 Peirce says his 1870 paper was the most important development in logic since the work of Boole.)

  Peirce's work on relations would in turn be enthusiastically embraced by E. Schröder, and would be the basis for Vol. III of Schröder's Algebra der Logik. Both Löwenheim and Skolem were well versed in Schröder's volumes. Löwenheim would follow this work on relations with his famous theorem on countable models in first-order languages, and Skolem would give this theorem its definitive proof. Skolem worked on decidability questions for first-order logic stemming from Schröder's work, and also recommended the use of first-order language (from Schröder's Vol. III) for the axiomatization of Zermelo's set theory. So one can actually trace a direct link from De Morgan's work to some of the highlights of modern logic.

De Morgan only sees the relevance of his ideas to the syllogism. Although he creates some of the most basic concepts of modern logic, by applying them only to syllogistic arguments he ends up with little credit for the development of modern logic.

CHAPTER 4

George Boole (1815–1864)

George Boole was a school teacher in Lincoln, England, when he began publishing, in 1840, respected papers in analysis in the Cambridge Mathematical Journal. His early (and primary) interests[1] were linear transformations, invariants, differential equations, and the calculus of variations. Starting to publish at the age of 25 is not particularly striking, until one realizes that Boole had never been a student in an institution of higher learning. He had simply taught himself all the higher mathematics, and much of the foreign languages, that he knew. For his 1844 paper on operational methods in differential equations he received a Royal Medal from the Royal Society.

One of the major influences on Boole early in his research career was the young Cambridge mathematician Duncan Gregory. Gregory stated three laws in 1839,

$$xy = yx \qquad x(u+v) = xu + xv \qquad x^m \cdot x^n = x^{m+n},$$

and said that these were all Euler needed to derive the binomial theorem (with fractional exponents). Gregory then used the three laws to justify applying the binomial theorem in his work with differential operators. In Boole's 1841 paper on linear DEs he says, without proof, that the three laws of Gregory also suffice to obtain the partial fraction decomposition of a rational function, and he applies this to the inverse of a linear differential operator with constant coefficients. Next, in Boole's prize winning paper of 1844, on differential and difference equations, he states the three laws as a basis for his work, inquiring if the third law might not be merely a necessity of notation.

The impact of Boole's early work with differential operators on his subsequent work with logic is rather clear. Replacing Gregory's third law with $x^n = x$ gives Boole's three laws of logic in 1847. However, Boole goes further by adding a *rule of inference* that we now call the *replacement rule* in equational logic, and by claiming that, in view of the first two of these laws, "all the processes of common algebra are applicable to the present system". What Boole actually uses is (with rare exceptions) just the equational part of common algebra. The laws of 1847 are clearly inadequate for what Boole does. But the point is that Boole (incorrectly) believes that the two laws give him the right to use the common (equational) algebra, and he immediately invokes that right.

Boole initiated a correspondence with De Morgan in 1842 that was to have a profound impact on Boole's life. De Morgan, nine years older than Boole, was highly regarded for his wide ranging intellectual writings, his lecturing expertise, and his popular texts in mathematics. Their early correspondence dealt with analysis, especially differential equations.

As mentioned in the section on De Morgan, by the Spring of 1847 each was engaged in writing a book on logic. Jevons and Venn would later claim that the equational approach to logic was a consequence of the discovery of the quantification of the predicate by De Morgan and Hamilton. All Boole says on this matter in his 1847 work is:

> In the spring of the present year my attention was directed to the question then moved between Sir W. Hamilton and Professor De Morgan; and I was induced by the interest which it inspired, to resume the almost-forgotten thread of former inquiries. It appeared to me

[1] Most of my colleagues think Boole was primarily a logician. But he was an analyst who suddenly acquired an interest in logic in 1847, worked on the subject for the next seven years, and then returned to his interests in analysis.

that, although Logic might be viewed with reference to the idea of quantity, it had also another and a deeper system of relations.

1. The Mathematical Analysis of Logic (1847)

The method of Boole's system is: (1) to translate propositions into equations in *the language of the algebra of numbers*, (2) to then apply the rules of the algebra of numbers, *slightly modified*, to derive new equations, and finally (3) to translate the new equations back into propositions in traditional logic.

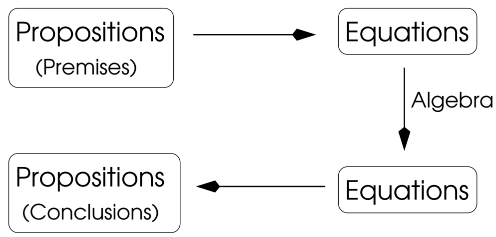


Fig. **34** Boole's Logical System

Boole's algebra of logic is equational logic applied to two situations: in the categorical logic it provides what is essentially **a calculus of classes**, and in the hypothetical logic it provides **a propositional calculus**. From the perspective of De Morgan's work, Boole has developed an equational calculus of compound names.

Boole's initial goal is simply to express all of the statements of traditional logic (categorical and hypothetical) as equations, and apply suitable algebraic transformations to the equations to derive the known valid arguments (the conversions, syllogisms, and hypothetical syllogisms) of logic.

Toward the end of writing this book Boole realizes that his algebra of logic is applicable to any finite collection of premises with any number of SYMBOLS. The traditional syllogisms would be just a small corner of this system, and would no longer merit the memory work on the classification and rules that had been staples for so many centuries. What was important to remember was how to translate between propositions and equations, and the rules of the algebra of logic. This would lead to the end of the dominance of Aristotelian logic—brought down by the writings of a self-taught English school teacher.

Boole's work has serious shortcomings. First he fails to provide clear definitions of most of the basic operations he uses to combine classes. Secondly, he claims, on rather meager evidence, that the laws of thought are indeed governed by the rules and processes of the algebra of numbers, with the *only* modification being the law $x^2 = x$. As mentioned in the introduction, the symbol x in this law behaves like a constant.[2]

Boole clearly defines only the first of the four operations *product, sum, difference,* and *quotient* that he applies to classes. The notation for these operations is borrowed from ordinary algebra, and he explains that the product xy refers to what we now call intersection. Addition is only defined in passing, when he discusses the distributive law $x(u+v) = xu+xv$. In this context the u and v will represent *disjoint* classes, and $u+v$ refers to their union. Yet $+$ will later appear in expressions where there is no reason to think that the terms represent disjoint classes. Minus is only explained

[2] We cannot substitute $x+x$ for x to derive $(x+x)^2 = x+x$; this would lead to $x+x = 0$, which Boole certainly does not accept.

for the expression $1-x$, which represents the complement[3] of the class represented by x. In the last chapter division is used in an essentially formal manner, with its interpretation depending entirely on applying the Expansion Theorem.

It is easy to imagine that Boole writes out the various equational laws now called the axioms, or defining laws, of Boolean algebra and uses them to analyze equations. This is not the case. Indeed he only has the following *three laws* in 1847 (page 17), clearly borrowed from his earlier work with differential operators:

$$\begin{aligned} x(u+v) &= xu+xv \\ xy &= yx \\ x^n &= x \end{aligned}$$

Fig. **35** Boole's Laws of 1847

At this point he claims that the first two laws justify the fact that "all the processes of common algebra" apply.[4] This claim is also clearly anchored in his work on differential operators.

Now let us turn to some of the details of his presentation. If one considers his successes with the calculus of differential operators it is perhaps not so unusual to find that Boole's 1847 system of logic is based on a study of **operators on classes**. He uses *elective* symbols x, y, etc., to denote the mental process of selecting, from any given class, only those elements which belong to the class determined by X, Y, etc. On page 5 we have

> Assuming the notion of a class, we are able, from any conceivable collection of objects, to separate by a mental act, those which belong to the given class, and to contemplate them apart from the rest. Such, or a similar act of election, we may conceive to be repeated.

The expression xy denotes: first select those elements that are in Y, and from the result those that are in X. The expression $x+y$ seems surely to denote: select those from X as well as those from Y, provided X and Y referred to disjoint classes. He first introduces the notation + in the setting of the distributive law (page 16):

> 1st. The result of an election is independent of the grouping or classification of the subject.
>
> Thus it is indifferent whether from a group of objects considered as a whole, we select the class X, or whether we divide the group into two parts, select the Xs from them separately, and then connect the results in one aggregate conception.
>
> We may express this law mathematically by the equation
>
> $$x(u+v) = xu+xv$$
>
> $u+v$ representing the undivided subject, and u and v the component parts of it.

In his 1854 system the symbols x, y, etc., would simply denote classes, and not the act of selection. There is no real change of content if we simply treat his elective symbols as symbols for classes in this discussion, so we will do just that.

After giving the three laws mentioned above he states that only *one axiom* is needed (page 18):

> [In view of the first two laws] all the processes of common algebra are applicable to the present system. The one and sufficient axiom involved in this application is that equivalent operations performed upon equivalent subjects provide equivalent results.

[3]Like De Morgan he introduces names for complements, but unlike De Morgan, his names for complements are compound names. The notations in Boole's book clash with that of De Morgan. De Morgan uses X and x for names of complementary classes, whereas Boole uses X and not-X. For the algebraic SYMBOLS that represent these classes Boole will use x and 1-x.

[4]For all but the last two chapters this means the usual polynomial algebra with the additional law $x^2 = x$. The last two chapters include power series and division. These laws are certainly not enough to justify polynomial algebra, and in his 1854 book the list increases from three to seven laws (but is still incomplete).

This axiom is a version of what is called the **Replacement Rule** in modern equational logic. This is not the only rule of inference that Boole needs for his logic—he also needs the **reflexive, symmetric, and transitive properties of equality**. Also he will use an **additively nonnilpotent property**, namely that $at = 0$ leads to $t = 0$ if a is a nonzero number.

Boole regards recognition of the above mentioned axiom as important for the foundations of logic, and he has a lengthy footnote chastising traditional logic for omitting this from its fundamental rules of reasoning regarding the principles of reduction that it uses (page 18):

> It is generally asserted by writers on Logic, that all reasoning ultimately depends on an application of the dictum of Aristotle, *de omni et nullo*. "Whatever is predicated universally of any class of things, may be predicated in like manner of any thing comprehended in that class." But it is agreed that this dictum is not immediately applicable in all cases, and that in the majority of instances, a certain previous process of reduction is necessary. What are the elements involved in that process of reduction? Clearly they are as much a part of general reasoning as the dictum itself.
>
> Another mode of considering the subject resolves all reasoning into an application of one or other of the following canons, viz.
>
> 1. If two terms agree with one and the same third, they agree with each other.
>
> 2. If one term agrees, and another disagrees, with one and the same third, these two disagree with each other.
>
> But the application of these canons depends on mental acts equivalent to those which are involved in the before-named process of reduction. We have to select individuals from classes, to convert propositions, &c., before we can avail ourselves of their guidance. Any account of the process of reasoning is insufficient, which does not represent, as well the laws of the operation which the mind performs in that process, as the primary truths which it recognizes and applies.
>
> It is presumed that the laws in question are adequately represented by the fundamental equations of the present Calculus. The proof of this will be found in its capability of expressing propositions, and of exhibiting in the results of its processes, every result that may be arrived at by ordinary reasoning.

With this preamble he proceeds to introduce the *minus* symbol to have a term for the class of 'not-X', namely $1 - x$, and then later it would (without explanation) appear in expressions such as $zy - y$. On page 20 we have:

> The class X and the class not-X together make the Universe. But the Universe is 1, and the class X is determined by the symbol x, therefore the class not-X will be determined by the symbol $1 - x$.

This does not read as if Boole is making a definition of $1 - x$, but rather as if Boole is proving that the symbol for not-X has to be $1 - x$.

To develop his system he first shows how to express the categorical Aristotelian propositions as equations:

A	All Xs are Ys	$xy = x$, or $x(1-y) = 0$
E	No Xs are Ys	$xy = 0$
I	Some Xs are Ys	$v = xy$
O	Some Xs are not Ys	$v = x(1-y)$

Boole's work seems to use the restricted semantics of SYMBOLS that we discussed in the Introduction, namely that the SYMBOLS may not be interpreted as the empty class or the universe. Under the restricted semantics the equation $v = xy$ guarantees that X and Y do have elements in common, namely those of V.

Regarding the strength of this equational foundation Boole says (page 22):

> The above equations [for the AEIO forms] involve the complete theory of categorical Propositions, and so far as respects the employment of analysis for the deduction of logical inferences, nothing more can be desired.

Then he gives some equational consequences in two cases (I and O) that he will need to make his analysis work properly (page 22):

> But it may be satisfactory to notice some particular forms deducible from the third and fourth equations [for the I and O forms] ...

These are (with the original equations in brackets)

		additional forms	
I	Some Xs are Ys	$v = vx$	$v(1-x) = 0$
	$[v = xy]$	$v = vy$	$v(1-y) = 0$
		$vx = vy$	
O	Some Xs are not Ys	$v = vx$	$v = v(1-y)$
	$[v = x(1-y)]$	$vx = v(1-y)$	$vxy = 0$

Fig. **36** Additional Equations for Particular Propositions

Boole comments on these additional equations (page 23):

> ... they give a precision and a definiteness to its conclusions, which could not otherwise be secured.

He means that he needs these variations to make things work.

Boole, like De Morgan, permitted the use of 'not-X' anywhere one could use an 'X', thus adding propositions such as

All not-Xs are not-Ys

to the repertoire. Following De Morgan let us call such statements **simple propositions** (Boole does not have a special name).

For the purpose of classification he treats 'not-X' simply as a term, so the AEIO classification scheme applies. Then 'All Xs are not-Ys' would be a universal positive statement (A), even though it is clearly equivalent to 'All Xs are not Ys', a universal negative statement (E).

In the chapter Of the conversion of Propositions Boole derives the classical rules for *conversion*. At the end of the chapter he proposes an extension of the work on conversion to cover all valid inferences from a single simple proposition, i.e., to cover what De Morgan calls *simple inferences* for simple propositions. The following is his system of three rules for this purpose (page 30):

> 1st. An affirmative Proposition may be changed into its corresponding negative (A into E, or I into O), and *vice versa*, by negation of the predicate.
>
> 2nd. A universal Proposition may be changed into its corresponding particular Proposition, (A into I, or E into O).
>
> 3rd. In a particular-affirmative, or universal-negative Proposition, the terms may be mutually converted.
>
> Wherein negation of a term is the changing of X into not-X, and *vice-versâ*, and is not to be understood as affecting the *kind* of the Proposition.
>
> Every lawful transformation is reducible to the above rules.

Boole does not substantiate his claim that the above rules are complete. He only provides a few lines of examples. However it is not difficult to check that he is indeed correct. We can write out these rules completely, where we use □ and ■ to denote a SYMBOL and its complement, and likewise for ◊ and ⧫. They can appear in any of the following four combinations, for any two symbols V_1 and V_2 for classes:

	□	■	◊	⧫
1.	V_1	not-V_1	V_2	not-V_2
2.	V_1	not-V_1	not-V_2	V_2
3.	not-V_1	V_1	V_2	not-V_2
4.	not-V_1	V_1	not-V_2	V_2

Fig. **37** Possible assignments of SYMBOLS

Boole's transformations then become the following, where an arrow indicates the direction of the transformation, and the rule used is written above the arrow:

All □s are ◊s	$\xleftrightarrow{\text{1st}}$	No □s are ⧫s
Some □ is ◊	$\xleftrightarrow{\text{1st}}$	Some □ is not ⧫
All □s are ◊s	$\xrightarrow{\text{2nd}}$	Some □ is ◊
No □s are ◊s	$\xrightarrow{\text{2nd}}$	Some □ is not ◊
Some □ is ◊	$\xleftrightarrow{\text{3rd}}$	Some ◊ is □
No □s are ◊s	$\xleftrightarrow{\text{3rd}}$	No ◊s are □s

Fig. **38** Boole's Transformations

By iterating these transformations one obtains the following diagram, and one can readily check that it is closed under the transformations:

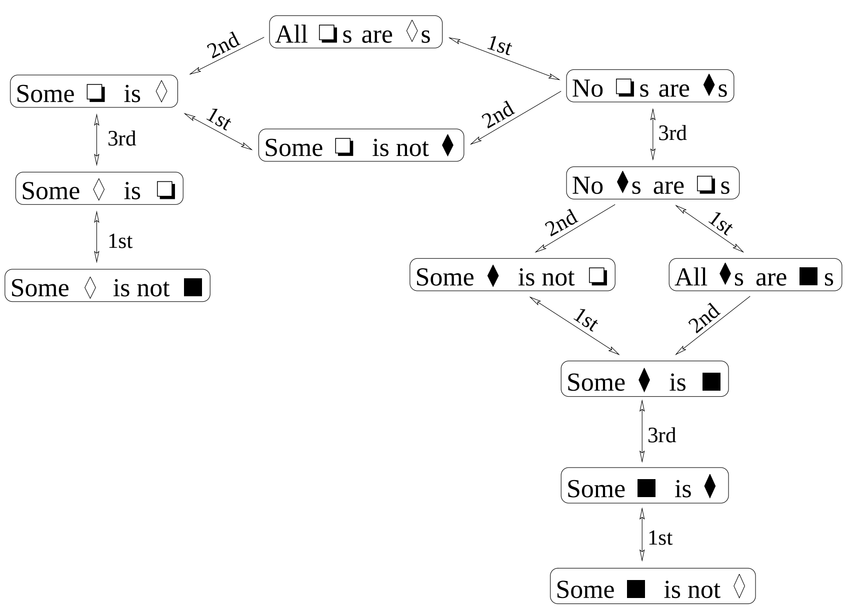


Fig. **39** Iterating Boole's Transformations

All four forms (AEIO) of simple propositions appear in this diagram. One can see that 12 propositions (including the given proposition) can be derived from either an A or an E statement, and 4 propositions from either an I or an O statement.

The rules of simple inference given by Boole yield valid inferences under the restricted semantics. One can check that indeed these rules will, by iteration, yield all simple inferences for simple propositions.

The chapter Of Syllogisms opens with the general strategy statement (pages 31–32):

> The equation by which we express any Proposition concerning the classes X and Y, is an equation between the symbols x and y, and the equation by which we express any Proposition concerning the classes Y and Z, is an equation between the symbols y and z. If from two such equations we eliminate y, the result, if it do not vanish, will be an equation between x and z, and will be interpretable into a Proposition concerning the classes X and Z. And it will then constitute the third member, or Conclusion, of a Syllogism, of which the two given Propositions are the premises.

The method of analyzing the syllogisms is to translate the two premises into two equations, say in the symbols x, y, z, with y representing the middle term, and then to write them as *linear*[5] equations $ay + b = 0$ and $cy + d = 0$, where y does not appear in the coefficients. Now Boole uses the fact that in ordinary algebra the resultant $ad - bc = 0$ is the most general equation that follows from eliminating y. Thus the crucial equational derivation

$$\begin{array}{rcl} ay + b & = & 0 \\ cy + d & = & 0 \\ \hline ad - bc & = & 0 \end{array}$$

is claimed to contain the essence of simple syllogistic reasoning. For example, for the premises of the AAA 1st Figure one has

$$\begin{array}{rcl} (1 - z)y & = & 0 \\ xy & = & x \\ \hline (1 - z)x & = & 0 \end{array}$$

yielding 'All Xs are Zs', as desired. But the method as stated does not always work, for consider the premises of the AAI 3rd Figure syllogism:

All Ys are Zs
All Ys are Xs

This leads to

$$\begin{array}{rcl} (1 - z)y & = & 0 \\ (1 - x)y & = & 0 \\ \hline 0 & = & 0 \end{array}$$

and thus not to the desired conclusion. So Boole says that for the system $ay = by = 0$ one should *expand* one of the equations. In the last example this would mean something like

$$\begin{array}{rcl} (1 - z)y & = & 0 \\ y & = & vx \\ \hline (1 - z)vx & = & 0 \end{array}$$

[5]All elective terms $\varphi(x, y, ...)$ can be reduced to polynomials with all the elective symbols to the first power as the elective symbols satisfy $x^2 = x$.

From this we can deduce $vx = vxz$, and as vx means 'some-X', it follows that we have 'Some X is Z'.

With this setup Boole claims that the resultant always provides the most general conclusion, and proceeds to list all of the forms and figures of the valid syllogisms, subdivided into four cases, depending on whether or not an auxiliary symbol v is used, and how it is used. Although he only illustrates the various cases with examples, indeed the method of Boole seems to work to derive most of the valid syllogisms. As mentioned in the Introduction he mishandles the 2nd Figure when the premises are AA. We will comment on this later.

There are two flaws in Boole's analysis of simple syllogisms. In ordinary algebra the condition $ad - bc = 0$ is indeed a necessary and sufficient condition for the linear system

$$\begin{aligned} ay + b &= 0 \\ cy + d &= 0 \end{aligned}$$

to have a nonzero solution for y. Boole evidently assumes this will carry over to the algebra of logic, where a nonzero y would correspond to a nonempty class.

But $ad - bc = 0$ is not the correct resultant in the algebra of logic. By 1854 Boole will have discovered his Elimination Theorem that shows the resultant should be the equation[6]

$$(1) \qquad \left(b^2 + d^2\right)\left[(a+b)^2 + (c+d)^2\right] \;=\; 0.$$

This, it turns out, is exactly what one needs for there to be a solution using modern semantics, that is, including the possibility that $y = 0$ or $y = 1$. Additional conditions are needed to guarantee that there is a solution that is not 0, or not 1.

In the particular linear equations that Boole was considering, obtained from his translation of the premises of a simple syllogism into equations, it turns out that the classical resultant $ad - bc = 0$ is actually a consequence of (1), but is not equivalent to it. However it was Boole's good luck that, concerning consequences that could be expressed as simple propositions, the two are equivalent. But they are not strong enough to yield all the correct syllogisms, namely when the strongest conclusion from universal premises is a particular proposition. Boole is able to carry out his analysis of simple syllogisms only by declaring that the cases where the resultant does not work require special attention.

The second flaw with Boole's analysis of syllogisms is an aesthetic flaw rather than one of mathematical substance. In the last example he derives the conclusion 'Some X is Z' from the equation $(1 - z)vx \;=\; 0$ by recalling the side condition that the combination vx means 'some X', that is, there is a side condition $vx \neq 0$. Bringing the side condition into play seems to detract from the purity of the equational algebraic treatment. Boole could have remedied this deficiency, using the restricted semantics, by writing out translation rules such as

$$\left.\begin{aligned} y &= vx \\ (1-z)vx &= 0 \end{aligned}\right\} \qquad \text{yield} \qquad \text{`Some X is Z'}$$

To keep the generality that he claims for his algebra of logic he would need to find an appropriate collection of such translation rules.

We will not pursue this direction further as there is a simpler translation that Boole overlooked, namely

$$\text{`Some X is Y'} \qquad \text{corresponds to} \qquad v \;=\; vxy.$$

[6]It is interesting to note that Boole does not inform the reader of his 1854 book of this 1847 error. When he analyzes the simple syllogisms in the 1854 book he first uses a more specific pair of equations, more closely tied to the form of the premises, and then he jumps over the elimination step that produces the resultant and immediately presents the reader with the solution (of the resultant equation) for the variable x.

Note that $v = vxy$ is a consequence of Boole's translation $v = xy$, and it still ensures that 'Some X is Y' holds under the restricted semantics. When translating an argument into equational form one needs to use a new SYMBOL v for each premiss that is a particular proposition, and such a SYMBOL is not to be used again in any other translation of the premises. However when translating from an equation back to ordinary language, the symbol v can be any SYMBOL x or its contrary $1 - x$. These are simply conditions on translation, and not on the algebraic manipulations.

For example from $x = xy$ one can derive $x = xxy$, and this corresponds to the simple inference: 'All X is Y' therefore 'Some X is Y'. For the 2nd Figure AA premises 'All X is Y' and 'All Z is Y' we have the equations $xy = x$ and $zy = z$. From these we can derive $(1-y)(1-x)(1-z) = 1-y$, yielding 'Some not-X is not-Y'. This is a valid syllogism, under the restricted semantics, that Boole missed in 1847. We will say more about this variation on Boole's method of handling particular propositions in the next section (on Boole's 1854 book) and in Appendix 4.

After finishing the simple syllogisms Boole turns to the discussion of hypothetical syllogisms in the chapter Of Hypotheticals and points out that they are of a totally different character than that of the categorical syllogisms. First he states the *constructive hypothetical*

If A is B, then C is D
But A is B, therefore C is D.

and the *destructive hypothetical*

If A is B, then C is D
But C is not D, therefore A is not C.

and says the following on page 48:

> If we examine either of the forms of conditional syllogism above given, we shall see that the validity of the argument does not depend upon any considerations which have reference to the terms A, B, C, D, considered as the representatives of individuals or of classes. We may, in fact, represent the Propositions A is B, C is D, by the arbitrary symbols X and Y respectively, and express our syllogisms in such forms as the following:
>
> If X is true, then Y is true
> But X is true, therefore Y is true.
>
> Thus what we have to consider is not objects and classes of objects, but the truths of Propositions, namely, of those elementary Propositions which are embodied in the terms of our hypothetical premises.

Having pointed out that hypothetical syllogisms indeed deal with a different species of proposition he then proceeds to set up his previous equational system for the hypotheticals by using an appropriate interpretation of the SYMBOLS, an interpretation that we can find in Whately's discussion of how to transform a hypothetical into a categorical.

> To the symbols X,Y,Z, representative of Propositions, we may appropriate the elective symbols x, y, z, in the following sense.
>
> The hypothetical Universe, 1, shall comprehend all conceivable cases and conjunctures of circumstances.
>
> The elective symbol x attached to any subject expressive of such cases shall select those cases in which the Proposition X is true, and similarly for Y and Z.

After explaining that $1 - x$ will refer to the situations where X is false, he gives a table to show how two SYMBOLS, and their complements, can divide up the universe (page 50):

> Thus if we associate the Propositions X and Y, the total number of conceivable cases will be found as exhibited in the following scheme.
>
> | | Cases. | | Elective expressions. |
> |---|---|---|---|
> | 1st | X true, Y true | ... | xy |
> | 2nd | X true, Y false | ... | $x(1-y)$ |
> | 3rd | X false, Y true | ... | $(1-x)y$ |
> | 4th | X false, Y false | ... | $(1-x)(1-y)$ |

The *cases* that Boole is referring to correspond to the terms obtained by multiplying together SYMBOLS and their complements such that each SYMBOL in the situation being discussed is mentioned exactly once in the term. e.g., $x(1-y)(1-z)$ is such a term for the symbols x, y, z. Such terms are later called *constituents* by Boole. xy refers to the collection of circumstances in which both X and Y are true, etc. Then he goes on to explain the fact that these cases divide up the universe corresponds to the sum of the expressions for the cases being 1.

Now Boole starts his propositional logic. The equation $x = 1$ will say that the proposition X is (in all cases) true, and $x = 0$ that it is (in all cases) false. One does the same more generally (page 51):

> And in every case, having determined the elective expression appropriate to a given Proposition, we assert the truth of that Proposition by equating the elective expression to unity, and its falsehood by equating the same expression to 0.

Then he gives examples:

X and Y are true	$xy = 1$
X and Y are false	$(1-x)(1-y) = 1$, or $x + y - xy = 0$
One or the other is true	$(1-x)(1-y) = 0$, or $x + y - xy = 1$

Then, for the first time in the book, he breaks into the full generality that his system is capable of (page 52):

> RULE. *Consider what are those distinct and mutually exclusive cases of which it is implied in the statement of the given Proposition, that some one of them is true, and equate the sum of their elective expressions to unity. This will give the equation of the given Proposition.*

This is clearly a receipe for writing out what we call the *disjunctive normal form.*[7]

Here is another sample from this chapter (page 55). Consider the case that X and Y are exclusive (i.e., disjoint), and we want the elective expression for the proposition that 'One or the other is true'. Then, as $xy = 0$ by assumption, the expression will be

$$x(1-y) + (1-x)y = 1$$

Multiplying this out and combining terms he obtains

$$x^2 - 2xy + y^2 = 1$$

and then by extracting the square root he has

$$x - y = \pm 1$$

[7]In applications Boole does not include in the sum any of the cases that he knows from the premises to be 0.

and this represents the actual case; for, as when X is true or false, Y is respectively false or true, we have

$$\begin{aligned} x &= 1 \quad \text{or} \quad 0 \\ y &= 0 \quad \text{or} \quad 1 \\ \therefore x - y &= 1 \quad \text{or} \quad -1 \end{aligned}$$

There will be no difficulty in the analysis of other cases.

Equipped with this equational propositional calculus he methodically works through the traditional hypothetical syllogisms. At the end of the chapter on the hypothetical he summarizes the situation with the two logics that he has been working with (page 59):

> The distinction [between the two logics] is real and important. Every Proposition which language can express may be represented by elective symbols, and the laws of combination of those symbols are in all cases the same; but in one class of instances the symbols have reference to collections of objects, in the other, to the truths of constituent Propositions.

This distinction seems to be over stated since in the propositional logic he is letting an elective symbol x refer to the *class* of situations in which the associated proposition X is true.

After finishing the chapter on the hypothetical, where Boole presents his first result for arbitrary expressions, he turns to the general study of logical expressions in the chapter titled Properties of Elective Functions. Elective functions are now called *terms* by logicians (see Appendix 1, for example). In this book Boole uses lower case Greek letters $\varphi, \psi, \ldots$ for his elective functions, but in his 1854 book he switches to Latin letters, with a preference for the letters f and V, with the letter t used for constituents. In our commentary, and in the appendices, we will use lower case Latin letters p, q, r, s, t for terms.

He opens with a remarkable leap into the use of power series:

> Since elective symbols combine according to the laws of quantity, we may, by Maclaurin's theorem, expand a given function $\varphi(x)$, in ascending powers of x, known cases of failure excepted. Thus we have
>
> $$\varphi(x) = \varphi(0) + \varphi'(0)x + \frac{\varphi''(0)}{1.2}x^2 + \&c., \ (44)$$
>
> Now $x^2 = x$, $x^3 = x$, &c., whence
>
> $$\varphi(x) = \varphi(0) + x\left\{\varphi'(0) + \frac{\varphi''(0)}{1.2} + \&c.\right\}, \ (45)$$
>
> Now if in (44) we make $x = 1$, we have
>
> $$\varphi(1) = \varphi(0) + \varphi'(0) + \frac{\varphi''(0)}{1.2} + \&c.,$$
>
> whence
>
> $$\varphi'(0) + \frac{\varphi''(0)}{1.2} + \frac{\varphi'''(0)}{1.2.3} + \&c. = \varphi(1) - \varphi(0).$$
>
> Substitute this value in for the coefficient of x in the second member in (45), and we have
>
> $$\varphi(x) = \varphi(0) + \left\{\varphi(1) - \varphi(0)\right\}x, \ (46)$$

He gives the following footnote to this last equation:

> Although this and the following theorems have only been proved for those forms of functions which are expansible by Maclaurin's theorem, they may be regarded as true for all forms whatever; this will appear from the applications. The reason seems to be that, as it is

only through the one form of expansion that elective functions become interpretable, no conflicting interpretation is possible.

Then he says, of the last equation above for $\varphi(x)$ (page 61):

> ... which we shall also employ under the form
>
> $$\varphi(x) = \varphi(1)x + \varphi(0)(1-x), \ (47).$$

This form is indeed, without exception, correct, and would be an important step to Boole's Elimination Theorem of 1854.

From this start Boole introduces his general tool for analyzing arguments, the *Expansion Theorem*, which says: given any term $p(x, y, \dots)$ one can, in a simple fashion, express it as a sum of terms in which desired SYMBOLS only occur in the special form of constituents. For example:

$$\begin{aligned} p(x,y) &= p(1,y)x + p(0,y)x(1-x), \qquad \text{and} \\ p(x,y) &= p(1,1)xy + p(1,0)x(1-y) + p(0,1)(1-x)y \\ &\qquad + p(0,0)(1-x)(1-y). \end{aligned}$$

Let us call the first expansion a *partial* expansion, and the second a *complete* expansion. Boole calls the coefficients $p(a, b)$ in the complete expansion the **moduli** of the term p. On page 61 we have

> PROP. 1. Any two functions $\varphi(x), \psi(x)$, are equivalent, whose corresponding moduli are equal.

Then he says on page 62 that this generalizes to the fact that any two terms are equivalent iff they have the same moduli.

Boole notes that the constituents $t_1, \dots, t_n$ for a given finite set of elective symbols satisfy the following:

$$\begin{aligned} t_i^2 &= t_i && \text{for all } i \\ t_i t_j &= 0 && \text{for } i \neq j, \text{ and} \\ 1 &= t_1 + \cdots + t_n. \end{aligned}$$

After the discussion of moduli and constituents Boole turns to his method of using complete expansions to interpret elective equations (page 64):

> We are now prepared to enter upon the question of the general interpretation of elective equations. For this purpose we shall find the following Propositions of the greatest service.
>
> PROP. 2. If the first member [the left side] of the general equation $\varphi(xy\dots) = 0$, be expanded in a series of terms, each of which is of the form at [t denotes a constituent], a being a modulus of the given function, then for every numerical modulus a which does not vanish, we shall have the equation
>
> $$at = 0,$$
>
> and the combined interpretations of these several equations will express the full significance of the original equation.

After explaining why this is true he continues with:

> ... whence if a_1 is a numerical constant which does not vanish,
>
> $$t_1 = 0$$
>
> and similarly for all the moduli which do not vanish. And inasmuch as from these constituent equations we can form the given equation, their interpretations will together express its entire significance.

Boole could have made Proposition 2 clearer by immediately saying $t = 0$ instead of $at = 0$. Since one can interpret, in the language of classes, what it means for a constituent t to be 0, namely a certain intersection of classes is empty, this gives a translation from equations to propositions about classes.

He continues (page 66) with another form of equation that he will frequently want to interpret.

> PROP. 3. If $w = \varphi(xy\,..)$, $w, x, y,$ .. being elective symbols, and if the second member be completely expanded and arranged in a series of terms of the form at, we shall be permitted to equate separately to 0 every term in which the modulus a does not satisfy the condition
>
> $$a^n = a,$$
>
> and to leave for the value of w the sum of the remaining terms.

His further results in this directions are (page 67):

> PROP. 4. The functions $t_1 t_2 \ldots t_r$ being mutually exclusive, we shall always have
>
> $$\psi(a_1t_1 + a_2t_2 \,..\, + a_rt_r) = \psi(a_1)t_1 + \psi(a_2)t_2 \ldots + \psi(a_r)t_r, \quad (63)$$
>
> whatever may be the values of $a_1 a_2 \ldots a_r$ or the form of ψ.
>
> ...
>
> PROP. 5. Whatever process of reasoning we apply to a single given Proposition, the result will either be the same Proposition or a limitation of it.

Throughout his final chapter On the Solution of Elective Equations Boole uses *division*, an operation that is missing in his previous chapters. The problem tackled in this chapter is of the following genre: Given an elective equation $\varphi(x, y) = 0$, solve for y in terms of x. Boole's first method is to express y as a linear combination of the x constituents, i.e.,

$$y = vx + v'(1 - x),$$

plug this back into the original equation $\varphi(x, y) = 0$ and find what restrictions this places on v, v'. So far this sounds reasonable. But consider the following from page 73:

> Had we expanded the original equation with respect to y only, we should have had
>
> $$\varphi(x\,0) + \{\varphi(x\,1) - \varphi(x\,0)\}y = 0;$$
>
> but it might have startled those who are unaccustomed to the processes of Symbolical Algebra, had we from this equation deduced
>
> $$y = \frac{\varphi(x\,0)}{\varphi(x\,0) - \varphi(x\,1)},$$
>
> because of the apparently meaningless character of the second member. Such a result would however have been perfectly lawful ... .

Boole would take this expression for y and expand it on x to obtain:

$$y = \frac{\varphi(1\,0)}{\varphi(1\,0) - \varphi(1\,1)}x + \frac{\varphi(0\,0)}{\varphi(0\,0) - \varphi(0\,1)}(1 - x)$$

Then, on page 74, he would explain the different possibilities for the coefficients:

> In the interpretation of any general solution of this nature, the following cases may present themselves.
>
> The values of the moduli $\varphi(0\,0)$, $\varphi(0\,1)$, &c. being constant, one or more of the coefficients of the solution may assume the form $\frac{0}{0}$ or $\frac{1}{0}$. In the former case, the indefinite symbol $\frac{0}{0}$ must be replaced by an arbitrary elective symbol v. In the latter case, the term, which is multiplied by $\frac{1}{0}$ (or by any numerical constant except 1), must be separately equated to 0, and will indicate the existence of a subsidiary Proposition.

Since Boole can express the categorical propositions as equations of the form $\varphi = 0$ he can use his Expansion Theorem to claim on page 77:

> *Thus all categorical Propositions are resolvable into a denial of the existence of certain compound classes, no member of one such class being a member of another.*

The compound classes referred to correspond to the constituents. Thus every categorical proposition in the symbols $x, y, \ldots$ is equivalent to a collection of assertions that the classes corresponding to certain constituents are empty.

The last topic of the chapter is Boole's method of using Lagrange multipliers to handle a system of equations.

Upon publication, in November of 1847, Boole's claim that all the processes of common algebra follow from the first two laws was immediately challenged in a letter from his friend Cayley. This must have caught Boole off-guard. After all, for nearly a decade these laws had been used in publications as a basis for work in the calculus of differential operators. Cayley pointed out that $xy = xz$ does not lead to $y = z$. This would lead to considerable soul searching by Boole, and a new foundation for his second book on logic.

Two years after Boole published this treatise he secured a professorship at the new Queen's College, Cork, Ireland, thanks in part to the support of De Morgan. He lived in Cork for the next (and last) 15 years of his life. After moving there he married Mary Everest, a niece of Colonel Everest of the Indian Survey, after whom the famous mountain is named, and they had a family of five daughters.

2. The Laws of Thought (1854)

In 1854 Boole published a thorough revision of his treatise on logic as the first part of a book on logic and probability. We will only be concerned with the first part. In the preface he says:

> The following work is not a republication of a former treatise by the Author, entitled, "The Mathematical Analysis of Logic." Its earlier portion is indeed devoted to the same object, and it begins by establishing the same system of fundamental laws, but its methods are more general, and its range of applications far wider. It exhibits the results, matured by some years of study and reflection, of a principle of investigation relating to the intellectual operations, the previous exposition of which was written within a few weeks after its ideas had been conceived.

In this work Boole spends much more time philosophizing on the subject of logic, and trying to find acceptable justifications for the rules and principles that he uses. In this he fails miserably—much of the time he only manages to convince the reader that the reader's doubts are well founded. His faulty arguments are based solely on a faith that the symbolical method used in ordinary algebra can be applied to logic. But some foundational matters are handled well, such as the relation between the algebra of numbers and the algebra of logic. On page 6 we have:

> There is not only a close analogy between the operations of the mind in general reasoning and its operations in the particular science of Algebra, but there is to a considerable extent an exact agreement in the laws by which the two classes of operations are conducted. Of course the laws must in both cases be determined independently; any formal agreement between them can only be established *a posteriori* by actual comparison. To borrow the notation of the science of Number, and then assume that in its new application the laws by which its use is governed will remain unchanged, would be mere hypothesis.

Then he goes on to point out the considerable extent to which the laws of logic and those of number overlap:

> Now the actual investigations of the following pages exhibit Logic, in its practical aspect, as a system of processes carried on by the aid of symbols having a definite interpretation, and

> subject to laws founded upon that interpretation alone. But at the same time they exhibit those laws as identical in form with the laws of the general symbols of algebra, with this single addition, viz., that the symbols of Logic are further subject to a special law (Chap. II.), to which the symbols of quantity, as such, are not subject.

The main achievement of the new book is the discovery of a procedure to eliminate SYMBOLS from a set of premises and arrive at the most general conclusion. It is perhaps not surprising that Boole lists this as the first requirement of a general method in logic (page 8):

> 1st. As the conclusion must express a relation among the whole or among a part of the elements involved in the premises, it is requisite that we should possess the means of eliminating those elements which we desire not to appear in the conclusion, and of determining the whole amount of relation implied by the premises among the elements which we wish to retain.

His system will be able to do this. Regarding the elimination problem he also says:

> It proposes not merely the elimination of one middle term from two propositions, but the elimination generally of middle terms from propositions, without regard to the number of either of them ...

In yet stronger form he says on page 10:

> Given a set of premises expressing relations among certain elements, whether things or propositions: required explicitly the whole relation consequent among *any* of those elements under any proposed conditions, and in any proposed form. That this problem, under all its aspects, is resolvable, will hereafter appear.

En route he takes a swat at Aristotle's logic (page 10):

> ... I would remark:—1st. That syllogism, conversion, &c., are not the ultimate processes of Logic.

Boole formulates the overview of his logic very simply (page 27):

> PROPOSITION I.
>
> *All the operations of Language, as an instrument of reasoning, may be conducted by a system of signs composed of the following elements, viz.:*
>
> 1st. *Literal symbols, as x, y, &c., representing things as subjects of our conceptions.*
>
> 2nd. *Signs of operation, as $+$, $-$, $\times$, standing for those operations of the mind by which the conceptions of things are combined or resolved so as to form new conceptions involving the same elements.*
>
> 3rd. *The sign of identity, $=$.*
>
> *And these symbols of Logic are in their use subject to definite laws, partly agreeing and partly differing from the laws of the corresponding symbols in the science of Algebra.*

Then he goes on to say that the logical SYMBOLS will now denote classes, and not selection processes as in the 1847 book (page 28):

> Let us then agree to represent the class of individuals to which a particular name or description is applicable, by a single letter, as x.

He will at times refer to them as *symbols*, or *literal symbols*, or when distinguishing their intended interpretation, as *logical symbols*, or *numerical symbols*. Then he explains the meaning of xy:

> Let it further be agreed, that by the combination xy shall be represented that class of things to which the names or descriptions represented by x and y are simultaneously applicable.

From this the laws $xy = yx$ and $x^2 = x$ easily follow. Boole's treatment of multiplication as representing intersection is quite satisfactory. The same cannot be said for his handling of addition.

The next topic is $+$ (which was barely defined in the 1847 text) on page 32:

> We are not only capable of entertaining the conceptions of objects, as characterized by names, qualities, or circumstances, ... but also of forming the aggregate conception of a group of objects consisting of partial groups, each of which is separately named or described. For this purpose we use the conjunctions "and," "or," &c. "Trees and minerals," "barren mountains, or fertile vales," are examples of this kind. In strictness, the words "and," "or," interposed between the terms descriptive of two or more classes of objects, imply that those classes are quite distinct, so that no member of one is found in another. In this and in all other respects the words "and" and "or" are analogous with the sign $+$ in algebra, and their laws are identical.

Boole could have been simpler and more direct. This seems to say that $x + y$ is to be used only when x and y represent disjoint classes. This will not be cleared up for another 30 pages, in Chapter V.

As aggregation is the meaning of $+$, on page 33 he gives the laws $x + y = y + x$ and $z(x + y) = zx+zy$, each justified by a single example in which the addition takes place between disjoint classes. Then he says:

> The above are the laws which govern the use of the sign $+$, here used to denote the positive operation of aggregating parts into a whole. But the very idea of an operation effecting some positive change seems to suggest to us the idea of an opposite or negative operation, having the effect of undoing what the former one has done.

With this he introduces the minus operation with the interpretation of $x-y$ being 'All x except y'. He states the restrictions he has in mind:

> Here it is implied that the things excepted form a part of the things from which they are excepted.

After 30 pages he returns to the issue of the requirement that x and y represent disjoint classes in the expression $x + y$. As this is one of the more contentious points of his development we will examine his presentation (page 66):

> The expression $x + y$ seems indeed uninterpretable, unless it be assumed that the things represented by x and the things represented by y are entirely separate; that they embrace no individuals in common.

This is in fact the first time Boole deals with the issue that the terms of his algebra of logic might not be interpretable. There was no discussion of this issue in the 1847 book. Perhaps it seems amazing that Boole still does not realize the simplicity of working with the union (as De Morgan did in 1847). But there is a clear reason. Boole has elegant principles that tie his algebra of logic to the *ordinary* number system, and union simply does not work in this setting.

So Boole is left with the situation that many expressions, including the most simple like $x + y$, can be meaningless in logic. He goes into the question of whether or not this means he has to restrict the application of his operations to situations for which the result of the operation is meaningful. He say that such a restriction would destroy the system (pages 66–67):

> The question then arises, whether it is necessary to restrict the application of these symbolical laws and processes by the same conditions of interpretability under which the knowledge of them was obtained. If such restriction is necessary, it is manifest that no such thing as a general method in Logic is possible.

Given that he wants to have a general logical system he poses the crucial question (page 67):

> On the other hand, if such restriction is unnecessary, in what light are we to contemplate processes which appear to be uninterpretable in that sphere of thought which they are designed to aid?

Then he points out how unusual this lack of meaningfulness is:

> ... this apparent failure of correspondency between process and interpretation does not manifest itself in the *ordinary* applications of human reason.

He continues to elaborate on these objections:

> There are perhaps many who would be disposed to extend the same principle to the general use of symbolical language as an instrument of reasoning. It might be argued, that as the laws or axioms which govern the use of symbols are established upon an investigation of those cases only in which interpretation is possible, we have no right to extend their application to other cases in which interpretation is impossible or doubtful, even though (as should be admitted) such application is employed in the intermediate steps of the demonstration only.

Finally he is ready to present his position on this issue, to conclude the till now cautious preparation of the defense of the correctness of his methods. In a single paragraph his defense suddenly turns into wishful thinking (pages 67–68):

> But the objection itself is fallacious. Whatever our *à priori* anticipations might be, it is an unquestionable fact that the validity of a conclusion arrived at by any symbolical process of reasoning, does not depend upon our ability to interpret the formal results which have presented themselves in different stages of the investigation.

This defense of Boole is no doubt influenced by the success of symbolic adventures in the development of calculus during the preceding two centuries. Continuing, Boole introduces the following hopeless principle for evaluating symbolic methods (page 69):

> A single example of reasoning, in which symbols are employed in obedience to laws founded upon their interpretation, but without any sustained reference to that interpretation, the chain of demonstration conducting us through intermediate steps which are not interpretable, to a final result which is interpretable, seems not only to establish the validity of the particular application, but to make known to us the general law manifested therein. No accumulation of instances can properly add weight to such evidence.

His final trump card in his rebuttal of the objections comes from the reliable standby, complex numbers (page 69):

> The employment of the uninterpretable symbol $\sqrt{-1}$, in the intermediate processes of trigonometry, furnishes an illustration of what has been said.

This is the end of his justification of using meaningless expressions.

Then he turns to the alternative interpretation of his expressions, one which always makes sense, namely one can let the literal symbols range over the numbers 0 and 1. But we have jumped over some important topics. Returning to the discussion of the fundamental operations that Boole has introduced, he continues with the discussion of the laws that govern them, giving (page 36):

> But instead of dwelling upon particular cases, we may at once affirm the general axioms:—
>
> 1st. If equal things are added to equal things, the wholes are equal.
>
> 2nd. If equal things are taken from equal things, the remainders are equal.
>
> And it hence appears that we may add or subtract equations, and employ the rule of transposition above given just as in common algebra.

Except for the reference to the role of transposition[8] this is just another version of his 'single axiom' of 1847, a form of the modern **Replacement Rule**.

For the basic laws he has expanded the three from 1847 to seven:

---

[8]Transposition means moving a term from one side of an equation to the other, accompanied by a change in sign of the term.

$$\begin{aligned} xy &= yx \\ x^2 &= x \\ x+y &= y+x \\ z(x+y) &= zx+zy \\ x(1-x) &= 0 \\ x-y &= -y+x \\ z(x-y) &= zx-zy \end{aligned}$$

Fig. **40** Boole's Laws of 1854

He returns to the idempotent law $x^2 = x$, and notes that among the numbers there are only two that satisfy it, namely 0 and 1. On page 37 he says:

> Hence, instead of determining the measure of formal agreement of the symbols of Logic with those of Number generally, it is more immediately suggested to us to compare them with symbols of quantity *admitting only of the values 0 and 1*. Let us conceive, then, of an Algebra in which the symbols x, y, z, &c. admit indifferently of the values 0 and 1, and of these values alone.

One is tempted to think Boole is speaking of a two-element algebra, perhaps a Boolean algebra or a Boolean ring. This is by no means the case. He is limiting the SYMBOLS to take on the values 0 and 1, but he means for the operations to be calculated as usual, in the ordinary number system.

This now leads to one of the most powerful principles in Boole's logic, and one for which no justification whatsoever is given. It was not mentioned in the 1847 book. We will call this **Boole's Rule of 0 and 1** (pages 37-38):

> The laws, the axioms, and the processes, of such an Algebra will be identical in their whole extent with the laws, the axioms, and the processes of an Algebra of Logic. Difference of interpretation will alone divide them. Upon this principle the method of the following work is established.

This is a very powerful principle as stated, and was never used with full force by Boole. For it implies that to check the validity of an argument of logic that has been translated into the equational form

$$p_1(\vec{x}) = 0, \ldots, p_k(\vec{x}) = 0 \qquad \therefore\ p(\vec{x}) = 0$$

it is necessary and sufficient to verify that the argument is correct in the ordinary numbers for each assignment of 0s and 1s to the x_i. This is very similar to the modern use of truth tables.

Boole uses this principle for two facts in his 1854 book:

- The Expansion Theorem follows because one can easily check that the 0,1 assignments make the two sides equal (in the ordinary numbers). This Theorem had been proved in the 1847 book by using a Maclaurin expansion. In this book the Maclaurin expansion proof survives in a footnote, with the explanation that it applied only to those terms that admitted a Maclaurin expansion.
- If $at = 0$, with a being a nonzero number and with t being a constituent, then $t = 0$ follows. This is used by Boole to show that any equation can be interpreted as a collection of constituents set equal to zero.

Once Boole has adopted this elegant principle the possibility of using Boolean algebra or Boolean rings is excluded because the Rule of 0 and 1 gives: $x + x = 0$ implies $x = 0$; and $x + x = x$ implies $x = 0$. Boole seems to have been blinded by the success of this rule, a rule that tied logic to ordinary arithmetic, and thus failed to come up with a simpler algebra of logic that would be developed by his successors.

Boole says that, when given equations obtained from logic, one can view them as numerical equations for which all the usual algebraic steps make sense, and at the end of the derivation one can switch back to the logical interpretation (page 70):

> *We may in fact lay aside the logical interpretation of the symbols in the given equation; convert them into quantitative symbols, susceptible only of the values 0 and 1; perform upon them as such all the requisite processes of solution; and finally restore them to their logical interpretation.* And this is the mode of procedure which will actually be adopted, ...

With the operations established, Boole turns on page 47 to the use of 1 for the Universe and 0 for Nothing, now treating Nothing as a class. He had already decided that his universe of discourse would be the "actual universe" (page 44). We have a curious remark in a footnote on page 50, where Boole is talking about the fact that the equation $x^3 = x$ does not have any interpretation in logic because it factors as $x(1-x)(1+x) = 0$; and the term $1 + x$

> . . . is not interpretable, because we cannot conceive of the addition of any class x to the universe 1 . . .

This suggests that one reason Boole did not discover a natural interpretation of $x + y$ for all classes x and y was that he fell victim to his choice of words for the operation, namely *addition*, which is in harmony with his notion of *aggregating*. These words suggest that whenever + has meaning it should allow some increase in the class, some augmentation of either class being added. Since 1 cannot be increased, Boole was at a loss for an interpretation.

In Chapter IV, Divisions of Propositions, Boole explains how to translate the categorical propositions into equations. He chooses equations from his 1847 work, but not the original ones.

		1854 versions	original 1847 versions
A	All Xs are Ys	$x = vy$	$xy = x$
E	No Xs are Ys	$x = v(1-y)$	$xy = 0$
I	Some Xs are Ys	$vx = vy$	$v = xy$
O	Some Xs are not Ys	$vx = v(1-y)$	$v = x(1-y)$

Fig. **41** Comparing 1854 and 1847 Translations

The SYMBOL v is to carry the meaning of 'some'. The v in the universal propositions is treated like any other SYMBOL, and one can derive the usual $x(1-y) = 0$ from the form $x = vy$ by eliminating v, so $x = vy$ yields at least as much information as in the original 1847 version. One does not derive anything that is incorrect from this translation of the universal propositions, using either the modern or restricted semantics.

In the case of the particular propositions the SYMBOL v is:

> . . . the symbol of a class indefinite in all respects but this, that it contains some individuals of the class to whose expression it is prefixed . . .

As mentioned in the Introduction, we will refer to this as the **side condition** on v. In Appendix 4 we will show that Boole could have replaced '$vx = vy$ + side condition on v' with the much simpler $v = vxy$. We need a bit of caution here when using the modern semantics.

Peirce (1880 [**16**], page XX) and later Schröder (1891, [**18**], page XX) will blast Boole for his handling of the particular categorical statements. Both will claim that it cannot be done with equations, and Schröder gives a proof of this fact. As Schröder notes in 1891, Vol. II of [**18**], page XX, the side condition can be simply formulated as $vx \neq 0$, and in the particular affirmative case Schröder replaces the combination '$vx = vy$ and $vx \neq 0$ with the simpler $xy \neq 0$, eliminating any need for a parameter.

In Appendix 4 we show that Boole was, with minor changes, correct. For either of the semantics we have discussed, modern or restricted, we can find simple equational techniques, using parameters, that capture the essence of particular propositions. The introduction of parameters for the universal propositions is wholly unnecessary.

In Chapter V, Principles of Symbolical Reasoning, Boole applies the Rule of 0 and 1 to justify *the expansion* (or *the development*) of a term. For example if $p(x)$ is a term in a single literal symbol we must have

$$p(x) \;=\; p(1)x \;+\; p(0)(1-x)$$

because the two sides agree numerically when x takes on the values 0 or 1. The same argument works for a term in several literal symbols. As before the *constituents* are the terms that are of the form of a product $\widehat{x}_1 \cdots \widehat{x}_n$, where each $\widehat{x}_i$ is either x_i or $1-x_i$. Thus the Expansion Theorem allows one to express any term $p(x_1, \ldots, x_n)$ as $\sum a_i t_i$, a sum of integer coefficients times constituents.

One of the expansions he gives is

$$x - y \;=\; x(1-y) \;-\; y(1-x)$$

and he remarks that this is generally uninterpretable in logic as (page 77):

> We cannot take, in thought, from the class of things which are x's and not y's, the class of things which are y's and not x's, because the latter class is not contained in the former.

Again Boole is a victim of his description of the operation—rather than try to extend the operations so that the laws are preserved he is concerned about an extension preserving the particular meaning he has attached to the symbol.

Nonetheless he says that if one derives an equation $x - y = 0$ then it has a perfectly legitimate interpretation, namely that the constituents in the expansion must be set to 0, i.e., one has $x(1-y) = 0$ and $(1-x)y = 0$. He says in summary (page 78):

> ... though *functions* do not necessarily become interpretable upon development, yet *equations* are always reducible by this process to interpretable forms.

He states the general theorem on the interpretation of an equation $V = 0$ in Chapter VI, Of Interpretation, on page 83:

> Rule.—*Develop the function V, and equate to 0 every constituent whose coefficient does not vanish. The interpretation of these results collectively will constitute the interpretation of the given equation.*

The main new result that Boole introduces in 1854 is the **Elimination Theorem** in Chapter VII, On Elimination, which shows how to find the most general equation (in some of the symbols) that can be obtained from a given equation. Later Schröder would call this Boole's main theorem. As an example suppose one is given an equation

$$p(x, y, z) = 0$$

in three symbols. Then the most general conclusion involving only the symbols x, z would be

$$p(x, 0, z)p(x, 1, z) = 0$$

The rule is simply to put 0's and 1's in place of the SYMBOLS you want to eliminate, in all possible ways, and then multiply these together and set the result equal to 0. If we have, for example,

$$p(x, y, z) \; = \; xyz \; + \; xy(1-z),$$

then this method would give

$$\Big(x0z + x0(1-z)\Big)\Big(x1z + x1(1-z)\Big) \; = \; 0,$$

which simplifies to

$$x(1-z) \; = \; 0,$$

and which we would interpret as 'All X is Z'.

Logical arguments usually have several premises, leading to several equations $V_1 = 0, \ldots, V_m = 0$. Chapter VIII, On the Reduction of Systems of Propositions, is devoted to methods for replacing such a system of equations by a single equation. For then one can apply the Elimination Theorem. Boole's favorite method is to use the single equation $V_1^2 + \cdots + V_m^2 = 0$. This leads to a number of examples of terms that are not interpretable in Boole's system.

Boole's work with division is essentially as in 1847, noting that in the expansion of the right hand side of an equation like

$$w \; = \; \frac{p(x_1, \ldots, x_n)}{q(x_1, \ldots, x_n)}$$

as a linear combination of constituents, $\sum a_i t_i$, the a_i can be one of four kinds: 0, 1, $\frac{0}{0}$, and *all others*. As before a coefficient of 0 means the constituent is omitted, a coefficient of 1 means the constituent is to be retained, a coefficient of $\frac{0}{0}$ is to be replaced by a new SYMBOL, and for the 'all others' case he deletes $a_i t_i$ from the expansion and sets $t_i = 0$ as a side constraint.

Boole's innovative approach to logic was not immediately appreciated. Whereas De Morgan's book gave thorough and clear explanations of material that was close to the traditional logic, Boole gave a brief and dubious justification for his approach, and later authors, especially Jevons, would describe Boole's treatment as obscure. The one exception was De Morgan who showed remarkable forsight in his Budget of Paradoxes:

> That the symbolic processes of algebra, invented as tools of numerical calculation, should be competent to express every act of thought, and to furnish the grammar and dictionary of an all-containing system of logic, would not have been believed until it was proved.
>
> ⋮
>
> The unity of the forms of thought in all applications of reason, however remotely separated, will one day be a matter of notoriety and common wonder; and Boole's name will be remembered in connection with one of the most important steps towards the attainment of this knowledge.

It is common to associate Boole's name with the laws of **Boolean algebra**, but it seems far more accurate to assign to him the near discovery of the laws of **Boolean rings**. Boole did not discover the laws of Boolean algebra because he was not working with Boolean algebra. His successors, especially Jevons, were responsible for this development. They could not make sense of what Boole was doing so they approached it from another direction, using the union operation described by De Morgan.

The book of 1854 emphasizes general results from the start, using the Rule of 0 and 1 to establish the Expansion Theorem, and then making heavy use of constituents and of the mysterious division. If one puts aside the steps involving division then all the ordinary algebraic manipulations make perfectly good sense when working with the interpretation of + as the *symmetric difference*, i.e., when working with the laws of Boolean rings.[9] However the methods of reduction must be considerably modified—see Appendix 1.

The natural extension of Boole's + chosen by Jevons was not symmetric difference, but *union*. Almost all the algebraic manipulations of Boole's book are correct[10] when one uses *union* for +. The problem with union is that some of the steps involving *minus* are not clear. If you want $1-x$ to denote the complement of x then what would $-x$ mean? For example how would one interpret the inference (on page 115) of $-xy-2x(1-y)=0$ from $xy-2x=0$? If one takes $-x$ as the complement of of x and $x-y$ as denoting the elements of x not in y then $x+(-y)$ is not in general equal to $x-y$ when working with union.

The Expansion and Elimination Theorems remain true *as stated* for Boolean rings, and a modified version of the Reduction theorem holds. The Rule of 0 and 1 holds with the integers $\mathbf{Z}$ replaced by the two-element Boolean ring $\mathbf{Z}_2$. (See Appendix 1 on Boolean rings.)

The situation with Boolean algebra is different as the axioms are different. With an appropriate change in the definition of a constituent the Expansion Theorem holds, essentially as stated. The Reduction Theorem is simpler, and the Elimination Theorem is essentially the same. The Rule of 0 and 1 holds with the integers $\mathbf{Z}$ replaced by the two-element Boolean algebra $\mathbf{B}_2$. (See Appendix 2 on Boolean algebra.)

Furthermore there is a third interpretation (due to Hailperin, 1976) under which Boole's results hold, and it captures the true spirit of the algebra of Boole. We discuss this in Appendix 3.

The dislike of Boole's successors (with the notable exception of John Venn) for uninterpretable terms led them to use union for +. This led to a thorough overhaul of the algebraic treatment, in many ways simplifying the presentation, and gave us the subject of Boolean algebra. A detailed version of the algebra of logic using union for + would be presented by Schröder in 1890, in the first volume of his **Algebra der Logik**,[11] up to and including the Elimination Theorem, with one exception. The main shortcoming of Schröder, and indeed everyone after Boole, was the complete omission of Boole's powerful Rule of 0 and 1.

After his 1854 book Boole returned to his original interests, and wrote successful textbooks on differential equations and difference equations. Repeatedly during the 15 years in Cork he would ask De Morgan if there were any possibility of obtaining an academic position in England, but nothing seemed to turn up. His new approach to logic was only beginning to be absorbed by others when unexpectedly, in the fall of 1864, he died—from a case of pneumonia that developed after being caught in an autumn shower while walking to his lectures, and then lecturing in wet clothes.

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[9] Lewis's 1918 A Survey of Symbolic Logic is rather confusing in his discussion of Boole's +. After explaining that Boole required x and y to represent disjoint classes in $x+y$ he goes on to say (page 53):

> $x+y$, then, symbolizes the class of things which are *either* members of x *or* members of y, but not of both.

If this means he thinks that Boole is working with the symmetric difference then Lewis has made a serious error.

[10] The main reason that so much of the algebra of classes in Boole's books is correct when viewed either as being carried out in the theory of Boolean algebras (with + being union) or in the theory of Boolean rings (with + being symmetric difference) is that he usually uses + between distinct constituents. The multiplications of Boolean algebras and Boolean rings are the same (intersection), and union agrees with symmetric difference on disjoint classes.

[11] Following ideas of Peirce from 1880, he takes '$\subseteq$' as primitive, and '=' as a defined relation. He carefully works out the equational theory entirely as derived *theorems*, including proofs that equality is reflexive, symmetric and transitive, and proving the replacement rule. The weak spot of Schröder's work is that he does not have a formal system to work with the primitive '$\subseteq$'.

CHAPTER 5

William Stanley Jevons (1835–1882)

Jevons had a passion for finding order in the world around him. Perhaps this was rooted in the losses sustained in his family during his childhood in Liverpool, England. Only six of the eleven children delivered by his mother survived. His mother died in 1845, and within a couple of years his oldest brother, Roscoe, fell victim to a devastating mental illness,[1] at the age of 18. His father's fortunes in the iron business deteriorated into bankruptcy in 1848.

At the age of 16 Jevons entered the University of London and studied for two years, with a particular interest in science. And during this time he would take extended walks through London to try to grasp the social structure of the city.

After the two years at the University of London he was offered a job in Sydney, Australia, as assayer to the new Royal Mint. With the encouragement of his father he accepted and set sail, first-class, on the three month voyage from Liverpool in the summer of 1854. Jevons had been in Australia only one year when his father died suddenly during a visit to Pisa, Italy. During Jevons' five year stay in Australia he took a deep interest in a wide range of scientific activities, keeping detailed records of the weather, studying the economy, the geology, the geography, the flora and the social structure.

He returned to England in 1859, taking a leisurely return through North America, where he visited an older brother who was trying log cabin pioneer life in Minnesota (it didn't last). Back in the University of London he continued the work on his academic degree (that he had left off five years before), concentrating on strengthening his scientific background, in particular the mathematics which he found difficult. He was convinced that mathematical skills were essential to a better understanding of the economy. During his years at the University of London he was greatly influenced by the vivacious and popular instructor Augustus De Morgan. In 1880 Jevons would write in the preface to **Studies in Deductive Logic**:

> There was never a greater teacher of mathematics than De Morgan; but from his earliest essay on the Study of Mathematics to his very latest writings, he always insisted upon the need of logic as well as purely mathematical training.
>
> ⋮
>
> My general indebtedness, both to those writings [on logic] and to his own unrivalled teaching, cannot be sufficiently acknowledged.

Jevons finished his M.A. in 1862. After an unsuccessful stint as a freelance writer in London he took a post as a tutor in Manchester in 1863. One of his early notable writing successes at Manchester was an article on the possibility that the British were squandering their economic future by their high consumption of cheap coal.

In 1866 he was appointed Professor, and within five years he would be publishing the reader-friendly books that would be used to educate nearly all British Empire students of elementary logic and economics for half a century.

[1] This was kept a family secret until 1955. Roscoe had to be cared for until his death in 1869. And a few months after this W.S.'s affectionate younger sister, Henrietta, fell victim to delusions which overwhelmed her for the remaining 40 years of her life.

In 1864 Jevons starts publishing works on logic that set out to modify the system of Boole so that all definitions, laws, and inferences are transparent as regards meaning. His first contribution is to replace Boole's partially defined + with *union*. This leads to the *Law of Unity* $\mathrm{A} + \mathrm{A} = \mathrm{A}$. Without explanation he keeps only the constant 0; perhaps writing $1 + 1 = 1$ would have been too controversial. He discards Boole's subtraction and division, the former because it is in conflict with the Law of Unity. To handle *complements* he adopts De Morgan's uppercase/lower case notation.

He emphasizes that his system has no connections with the algebra of number, that the interpretations of his expressions are all simple, natural items in logic. In 1869 he changes the + to $\cdot|\cdot$ to emphasize the non-numerical nature of the operation. His primary method of inference, the method of "indirect inference", is to make a list of all constituents involving the SYMBOLS of the premises, eliminate those that contradict any premise, and use the remaining constituents to derive conclusions. This is a tedious process, and between 1864 and 1869 Jevons had several ideas to expedite the work, culminating in a "machine capable of reasoning".

As a further effort to eliminate any mystery he attempts to give a complete list of the axioms and rules of inference that are needed. If one combines his secondary remarks with his primary list of these items, and if one ignores his requirements that the symbols A, etc., and their complements a, etc., be non-zero (see item 7 of §116 of **Pure Logic**), then, with the exception of the associative laws, he succeeds. This would make Jevons the first person to formulate an (essentially complete) equational logic, and it would be for Boolean algebra.

Jevons main contribution to the equational approach to logic would be his interpretation of + and the explicit presentation of the axioms and rules. His method of indirect inference would be considered primitive and tedious by his successors. And they would find fault with the lack of symbols for the universe and the operation of complementation, and the failure to carry over some theorems of Boole that were true in his system, in particular the expansion and elimination theorems. As mentioned at the end of the last chapter, all of Boole's successors, including Jevons, avoided mentioning the Rule of 0 and 1.

Jevons found teaching quite stressful, and by 1876 he was able to move to the University of London with a post requiring little lecturing, and in 1881 he even resigned from this to devote himself to writing, especially on his master treatise on economies. But the next summer, on a holiday outing, he drowned at the age of 46. There was some speculation that his poor health had contributed to this accident.

1. Pure Logic (1864)

In 1864 Jevons publishes[2] his first attempt to put Boole's equational work on a solid footing, without the mysterious mathematical trappings. He claims that the **quantification of the predicate** was the key ingredient that initiated the (equational logic) revolution against Aristotle's system:

> **139.** It is only of very late years that the imperfection of the ordinary proposition has been properly pointed out. It is the discovery of the so-called *quantification of the predicate* which has reduced the proposition to the form of a convertible equation, and opened out to logic an indefinite field of improvement.
>
> **140.** Professor Boole's system, first published in his *Mathematical Analysis of Logic*, in 1847, involves this newly discovered quantification of the predicate.

As we noted in the section on Boole's work, it is not clear that the quantification of the predicate played *any* role in Boole's work. Boole simply took the traditional forms of logic and converted them to equations.

[2]Our source for this section is **Pure Logic and Other Minor Works, 1971**.

Jevons wants to divorce the algebra of logic from the marriage with the ordinary number system that had been forged by Boole. Hence the 'Pure' in the title Pure Logic. And, to reinforce the point that logic can be treated without reference to quantity, Jevons differs with Boole on how to interpret the SYMBOLS. He maintains that they should be taken as **intensive** rather than extensive, running counter to British tradition. This choice of intensive rather than extensive mode explains the second part of his title,[3] for in the opening paragraph of the book he says:

> It is the purpose of this work to show that Logic assumes a new degree of simplicity, precision, generality, and power, when comparison in quality is treated apart from any reference to quantity.

He says that the intensive mode is more in touch with the common language. However his system is just as easily understood in the **extensive** mode, and we will view it in that manner. Admitting quantity in the context of classes in no way compromises Jevons program to avoid tying logic to the concept of quantity as represented by numbers. And even Jevons will eventually slip into using the extensive mode.

With this understanding we first summarize the definitions, axioms, and rules of inference Jevons uses, followed by a discussion of these items. In each case we will state a name if Jevons provides one. First we give the definitions:

Expression	Meaning	Name	Page
AB	intersection	**combination**	15
A + B	union	**plural term**	24
not-A	complement of A	**contrary of A**	30
a	complement of A	**contrary of A**	30
0	empty class	**excluded from thought**	31

Fig. **42** Definitions of Jevons

The members of a *plural term* are called **alternatives**. Jevons uses the operator "not-" quite sparingly, preferring the De Morgan notation. The clumsiest feature of his definitions is his introduction of 0, where he seems to be unsure of what is going on—we will discuss this below.

Next we look at the equations that Jevons says hold in general. We will call them his *axioms*:

Equation	Name	Page
A = A	**Postulate**	11
AA = A	**Law of Simplicity**	16
AB = BA		17
B + C = C + B		24
A(B + C) = AB + AC		25
A + A = A	**Law of Unity**	25
B + BC = B		25
not-not-A is A		30
contrary of BC is Bc + bC + bc		30
contrary of A + B + C is abc		30
Aa = Aa.0		31
Aa = 0	**Law of Contradiction**	31
0.0 = 0		32
0 + 0 = 0		32
A = AB + Ab	**Law of Duality**	34

[3]The full title of this work is: **Pure Logic, or the Logic of Quality apart from Quantity, With Remarks on Boole's System and on the Relation of Logic and Mathematics.**

Fig. **43** Axioms of Jevons in 1864

The five axioms that have names are the ones that he puts in his summary as the primary ones, the others *presumed* to be derivable.

Finally we want to list the five rules of inference that Jevons uses in this work. We will label them as R1–R5. A modern name for each rule is given in parentheses. Jevons' names for R2, R4, and R5, are given in bold type. (He does not have names for the others.)

R1. (Symmetric Rule for Equality) (page 10)

> The propositions $\mathrm{A} = \mathrm{B}$ and $\mathrm{B} = \mathrm{A}$ are the same statement; ...

This is not exactly how we would formulate this rule. Rather we would say that one can infer $\mathrm{B} = \mathrm{A}$ from $\mathrm{A} = \mathrm{B}$.

R2. (Transitive Rule for Equality) [**Law of Sameness**] (page 12)

> **25.** It is in the nature of thought and things that *things which are same as the same thing are the same as each other.*
>
> ⋮
>
> From $\mathrm{A} = \mathrm{B}$, $\mathrm{B} = \mathrm{C}$, which are the same in the member B, we may form the new proposition, $\mathrm{A} = \mathrm{C}$.

The modern reading is: from $\mathrm{A} = \mathrm{B}$ and $\mathrm{B} = \mathrm{C}$ one can infer $\mathrm{A} = \mathrm{C}$. In this work of Jevons it is maintained that the Law of Sameness is the primary rule. This will change in his next work (in 1869).

R3. (First version of: Replacement for Intersection) (page 17)

> *Same terms combined with same terms give same combined terms.*
>
> Thus, since $\mathrm{A} = \mathrm{A}$ and $\mathrm{B} = \mathrm{B}$, therefore $\mathrm{AB} = \mathrm{BA} = \mathrm{AB}$.

This particular version of replacement has the commutative law built in.

R4. (Second version of: Replacement for Intersection) [**Law of Same Parts and Wholes**] (page 17)

> 45. *Same terms being combined with both members of a premise, the combinations may be stated as same in a new proposition which will be true with the premise.*
>
> ⋮
>
> Thus, from $\mathrm{A} = \mathrm{B}$ we may infer $\mathrm{AC} = \mathrm{BC}$ by combining C with each of A and B.

R5. (Replacement) [**Substitution**] (page 18)

> *For any term, or part term, in one premise, may be [Jevons] substituted its expression (§29) in other terms.*
>
> In short, the two members of any premise may be used indifferently one in place of the other, wherever either occurs.

The first two rules that have bold-faced names are the ones Jevons considers primary, and the others derivable.

The modern version of the **Replacement Rule** says that if one is given an equation $p = q$ and p occurs as a subterm of s, something we can describe by writing $s[p]$, then one can replace the specified occurrence of p by q and obtain the equation $s[p] = s[q]$.

The modern meaning of the **Substitution Rule** in equational logic is that if one is given an equation $r(x_1, \ldots, x_n) = s(x_1, \ldots, x_n)$ and terms $t_1, \ldots, t_n$, then one can infer $r(t_1, \ldots, t_n) =$

$s(t_1, \ldots, t_n)$, the equation that results by the *simultaneous and uniform substitution* of the t_i for the x_i.

Jevons summarizes the above:

> **109.** The following are the chief laws or conditions of logic:—
>
> *Condition or postulate.* The meaning of a term must be same throughout any piece of reasoning; so that $\mathrm{A} = \mathrm{A}$, $\mathrm{B} = \mathrm{B}$, and so on.
>
> *Law of Sameness.*
> $$\mathrm{A} = \mathrm{B} = \mathrm{C}; \text{ hence } \mathrm{A} = \mathrm{C}$$
> *Law of Simplicity.*
> $$\mathrm{AA} = \mathrm{A},\ \mathrm{BBB} = \mathrm{B}, \text{ and so on.}$$
> *Law of Same Parts and Wholes.*
> $$\mathrm{A} = \mathrm{B}; \text{ hence } \mathrm{AC} = \mathrm{BC}.$$
> *Law of Unity.*
> $$\mathrm{A} + \mathrm{A} = \mathrm{A},\ \mathrm{B} + \mathrm{B} + \mathrm{B} = \mathrm{B}, \text{ and so on.}$$
> *Law of Contradiction.*
> $$\mathrm{A}a = 0,\ \mathrm{AB}b = 0, \text{ and so on.}$$
> *Law of Duality.*
> $$\begin{aligned} \mathrm{A} &= \mathrm{A}(\mathrm{B} + b) = \mathrm{AB} + \mathrm{A}b \\ \mathrm{A} &= \mathrm{A}(\mathrm{B} + b)(\mathrm{C} + c) \\ &= \mathrm{ABC} + \mathrm{AB}c + \mathrm{A}b\mathrm{C} + \mathrm{A}bc, \text{ and so on.} \end{aligned}$$
> It seems likely that these are the primary and sufficient laws of thought, and others only corollaries of them.
>
> ⋮
>
> The Laws of Simplicity, Unity, Contradiction, and Duality furnish the universal premises of reasoning. The Law of Sameness is of altogether a higher order, involving inference, or the Judgement of Judgements.

His treatment of the equivalence properties of equality, i.e., the **reflexive, symmetric, and transitive** properties, seems curious from a modern viewpoint. In §24 he refers to $\mathrm{A} = \mathrm{A}$ as a "useless *Identical* proposition." In §10 he states that $\mathrm{A} = \mathrm{B}$ is the same as $\mathrm{B} = \mathrm{A}$, but he does not seem to realize that this is a rule of inference as it is missing from his summary given above. And he attaches an extraordinary importance to the transitive property. In the context of discussing the equational form of propositions versus Aristotle's forms he says:

> **138.** ... that *reasoning from same to same things may be detected as the fundamental principle of all the sciences*, we need have no hesitation in treating the equation as the true proposition, and Aristotle's form as an imperfect proposition.
>
> It is then the Law of Sameness, not the *dictum* of Aristotle, which governs reason.

Conspicuously missing from the above summary are the **commutative laws**, **distributive laws**, **associative laws**, any version of **De Morgan's laws**, as well as the **absorption**[4] laws $\mathrm{A} + \mathrm{AB} = \mathrm{A}$ and $\mathrm{A}(\mathrm{A} + \mathrm{B}) = \mathrm{A}$. Also his summary fails to fully include the **replacement rule** that he had formulated quite clearly:

> For any term, or part-term, in one premise, may be substituted its expression in other terms.

We see part of this in his Law of Same Parts and Wholes, but he is missing the important rules: $\mathrm{A} = \mathrm{B}$ implies $\mathrm{A} + \mathrm{C} = \mathrm{B} + \mathrm{C}$, and $\mathrm{A} = \mathrm{B}$ implies not-A= not-B. By 1869 he will realize the central

[4]In 1880 Peirce attributes both absorption laws to Grassmann and Schröder, without any reference to the earlier work of Jevons.

role played by replacement, as expressed in his substitution rule, and will claim that this rule is the source of all correct reasoning.

Jevons' treatment of 0 also leaves much to be desired (page 9):

> The meaning of 0, whatever it exactly be, may also be expressed in words.
>
> ⋮
>
> **92.** Let us denote by the term or mark 0, combined with any term, that this is contradictory, and thus excluded from thought. Then $\mathrm{A}a = \mathrm{A}a.0$, $\mathrm{B}b = \mathrm{B}b.0$, and so on. For brevity we may write $\mathrm{A}a = 0$, $\mathrm{B}b = 0$. Such propositions are tacit premises of all reasoning.
>
> ⋮
>
> **94.** The term 0, meaning *excluded from thought*, obeys the laws of terms.
>
> $$0.0 = 0 \qquad 0 + 0 = 0,$$
>
> otherwised expressed:—What is excluded *and* excluded is excluded—What is excluded *or* excluded is excluded.

Missing from Jevons' system is a formulation of the laws

$$\mathrm{A}.0 = 0 \qquad \mathrm{A} + 0 = \mathrm{A}$$

although one does see them used in practice. Regarding the latter there is a curious argument:

> **96.** *In a plural term of which not all the alternatives are contradictory, the contradictory alternative or alternatives must be excluded from notice.*
>
> If for instance $\mathrm{A} = 0 + \mathrm{B}$, we may infer $\mathrm{A} = \mathrm{B}$, because A if it be 0 is excluded; and if it be such as we can desire knowledge of, it must be the other alternative B.

Let us look at the **indirect inference** method of Jevons to determine the consequences of a given set of equations. Suppose we are given k equations, say $p_1 = q_1, \ldots, p_k = q_k$, in the SYMBOLS A,B,C. The first step is to write out the complete list of constituents[5] in the SYMBOLS A,B,C:

ABC	aBC
ABc	aBc
AbC	abC
Abc	abc

Then one examines all equations $tp_j = tq_j$ resulting from multiplying the premises $p_j = q_j$ by the constituents t, i.e., with our example of 3 SYMBOLS one is to examine the list of $8k$ equations

$$\begin{array}{rcl} \mathrm{ABC}p_1 & = & \mathrm{ABC}q_1 \\ \mathrm{ABC}p_2 & = & \mathrm{ABC}q_2 \\ & \vdots & \\ abcp_k & = & abcq_k \end{array}$$

In each of these equations $tp_j = tq_j$ one has each of the terms tp_j and tq_j simplifying to one of two possibilities, namely 0 or t. Thus each of these equations simplifies to one of four forms, labeled as follows:

$t = t$	**included subject**	of the jth equation
$0 = 0$	**excluded subject**	of the jth equation
$t = 0$	**contradiction**	to the jth equation
$0 = t$	**contradiction**	to the jth equation

[5]Jevons did not use the name *constituents*, or any other special name, for these combinations.

Each t that appears in a contradiction is to be *struck off the list* of constituents. Once this is finished for all equations and all constituents then we say the following for each constituent t remaining on the list:

t	is an	**included subject**	if t is an included subject for some $p_j = q_j$
t	is an	**excluded subject**	if t is excluded subject for all $p_j = q_j$

Let us suppose that the above procedure has been carried out and that the remaining constituents are $t_1, \ldots, t_n$. We will refer to this as the *shortened* list of constituents. Then to find out what the premises say about a given term s, assumed to be a SYMBOL or a product of SYMBOLS, let $t_{i_1}, \ldots, t_{i_m}$ be the constituents in the shortened list that include all the SYMBOLS of s. Then

$$s = t_{i_1} + \cdots + t_{i_m}$$

follows from the premises.

Jevons also has rules for simplifying the right hand side of this equation, the first being to use the Law of Duality to combine terms, and the second is to note that if some part s_i of a t_i on the right only occurs in the one term t_i in the shortened list, then one can replace t_i by s_i.

There are a number of examples that Jevons looks at, the first being to consider the consequences of the single premise A = BC. In the following table we give the details of striking off the constituents:

	Multiply A=BC by a constituent	and	Simplify
ABC	ABC**A** = ABC**BC**		ABC = ABC
~~ABc~~	ABc**A** = ABc**BC**		ABc = 0
~~AbC~~	AbC**A** = AbC**BC**		AbC = 0
~~Abc~~	Abc**A** = Abc**BC**		Abc = 0
~~aBC~~	aBC**A** = aBC**BC**		0 = aBC
aBc	aBc**A** = aBc**BC**		0 = 0
abC	abC**A** = abC**BC**		0 = 0
abc	abc**A** = abc**BC**		0 = 0

Fig. **44** The Jevons Procedure

The above process leads to the following constituents that do not appear in contradictions:

ABC	included subject
aBc	excluded subject
abC	excluded subject
abc	excluded subject

Thus he has for the SYMBOL b the following consequence of AB = C :

$$b = ab\mathrm{C} + abc = ab(\mathrm{C} + c) = ab,$$

which translates into "All b is a". And for the term aB he has

$$a\mathrm{B} = a\mathrm{B}c\ .$$

As Bc can only occur in the constituent aBc in the shortened list of constituents, one can replace aBc by Bc to obtain

$$a\mathrm{B} = \mathrm{B}c\ .$$

This, then, is the method of Jevons, and he gives his assessment of how it compares to Boole's method:

> **173.** Compared with Professor Boole's system, in its mathematical dress, this system shows the following advantages:—
>
> 1. Every process is of self-evident nature and force, and governed by laws as simple and primary as those of Euclid's axioms.
>
> 2. The process is infallible, and gives no uninterpretable or anomalous results.
>
> 3. The inferences may be drawn with far less labour than in Professor Boole's system, which generally requires a separate computation and development for each inference.
>
> **174.** So long as Professor Boole's system of mathematical logic was capable of giving results beyond the power of any other system, it had in this fact an impregnable stronghold. Those who were not prepared to draw the same inferences in some other manner could not quarrel with the manner of Professor Boole. But if it be true that the system of the foregoing chapters is of equal power with Professor Boole's system, the case is altered. There are now two systems of notation, giving the same formal results, one of which gives them with self-evident force and meaning, the other by dark and symbolic processes. The burden of proof is shifted, and it must be for the author or supporters of the dark system to show that it is in some way superior to the evident system.
>
> **175.** It is not to be denied that Boole's system is consistent and perfect within itself. It is, perhaps, one of the most marvellous and admirable pieces of reasoning ever put together. Indeed, if ... the chief excellence of a system is in being *reasoned* and consistent within itself, then Professor Boole's is nearly or quite the most perfect system ever struck out by a single writer.
>
> **176.** ... Professor Boole's system is Pure Logic fettered with a condition which converts it from a purely logical into a numerical system.

After this comparison of his system with Boole's he goes on to list four objections to Boole's system:

First Objection

> **177.** *Boole's symbols are essentially different from the names or symbols of common discourse—his logic is not the logic of common thought.*

Jevons is mainly objecting to Boole's use of $+$ to express the connective "or" for disjoint classes, whereas the common use of "or" would allow overlapping classes. Jevons strongly feels that the inclusive definition of "or" is *the* correct interpretation of $+$ in logic.

Second Objection

> **184.** *There are no such operations as addition and subtraction in Pure Logic.*

This seems to be primarily an objection to Boole's use of *subtraction*, e.g., he says that from $\mathrm{A}+\mathrm{B}+\mathrm{C}=\mathrm{A}+\mathrm{D}+\mathrm{E}$ in logic one cannot subtract A and conclude $\mathrm{B}+\mathrm{C}=\mathrm{D}+\mathrm{E}$. The problem is simply that Jevons assumes his interpretation of $+$ is the one and only correct one. One can only wonder how he would have reacted to being shown the symmetric difference. But the symmetric difference did not come to the attention of logicians until the 1900s.

Third Objection

> **193.** My third objection to Professor Boole's system is that *it is inconsistent with the self-evident law of thought, the Law of Unity* $(\mathrm{A}+\mathrm{A}=\mathrm{A})$.
>
> ... It is surely self-evident, however, that $x+x$ is equivalent to x alone ... it is apparent that the process of subtraction in logic is inconsistent with the self-evident Law of Unity.

Again we see that Jevons is guilty of only being able to imagine one interpretation of his symbol $+$, this interpretation being a consequence of his choice of the *usual* definition of "or". Again, had someone discovered the symmetric difference at that time, much of this polemic would never have appeared.

Fourth Objection

197. The last objection that I shall at present urge against Professor Boole's system is, that *the symbols* $\frac{1}{1}$, $\frac{0}{0}$, $\frac{0}{1}$, $\frac{1}{0}$, *establish for themselves no logical meaning, and only bear a meaning derived from some method of reasoning not contained in the symbolic system.* The meanings, in short, are those reached in the self-evident indirect method of the present work.

⋮

199. ... Professor Boole's system, then, as regards the symbol $\frac{0}{0}$, is not the system bestowing certain knowledge; it is, at most, a system pointing out truths which, by another intuitive system of reasoning, we may know to be certainly true.

⋮

202. The correspondence of these obscure forms with the self-evident inferences of the present system is so close and obvious, as to suggest irresistibly that Professor Boole's operations with his abstract calculus of 1 and 0, are a mere counterpart of self-evident operations with the intelligible symbols of pure logic.

⋮

Boole's system is like the shadow, the ghost, the reflected image of logic, seen among the derivatives of logic.

⋮

205. ... these errors scarcely detract from the beauty and originality of the views he laid open. Logic, after his work, is to logic before his work, as mathematics with equations of any degree are to mathematics with equations of one or two degrees. He generalized logic so that it became possible to obtain any true inference from premises of any degree of complexity, and the work I have attempted has been little more than to translate his forms into processes of self-evident meaning and force.

So, after all his complaints about Boole's work, we find that he still has the greatest admiration for it. It is remarkable in looking through this text to see how infrequently 0 appears in Jevons equations. He much prefers that his equations be of the form "term = term", with both left and right sides being different from 0. This is quite different from Boole, who preferred to put his equations in the form "term = 0".

2. The Substitution of Similars (1869)

In this tract, subtitled **The True Principle of Reasoning**, Jevons says that Aristotle's *dictum* was appropriate to the traditional logic, but since the upheaval in the foundations of logic a new fundamental principle is needed to bring clarity:

7. In the lifetime of a generation still living the dull and ancient rule of authority has thus been shaken, and the immediate result is a perfect chaos of diverse and original speculations. Each logician has invented a logic of his own, so marked by peculiarities of his individual mind, and his customary studies, that no reader would at first suppose the same subject to be treated by all ... Modern logic has thus become mystified by the diversity of views, and by the complication and profuseness of the formulae invented by the different authors named. The quasi-mathematical methods of Dr. Boole especially are so mystical and abstruse, that they appear to pass beyond the comprehension and criticism of most other writers, and are calmly ignored. No inconsiderable part of a lifetime is indeed needed to master thoroughly the genius and tendency of all the recent English writings on Logic, and we can scarcely wonder that the plain and scanty outline of Aldrich, or the sensible but unoriginal elements of Whately, continue to be the guides of a logical student, while the works of De Morgan or of Boole are sealed books.

⋮

10. During the last two or three years the thought has constantly forced itself upon my mind, that the modern logicians have altered the form of Aristotle's proposition without making any corresponding alteration in the *dictum* or self-evident principle which formed the fundamental postulate of his system.

⋮

11. But recent reformers of logic have profoundly altered our view of the proposition. They teach us to regard it as an equation of two terms ... Does not the *dictum*, in short, apply in both directions, now that the two terms are indifferently subject and predicate?

⋮

14. I am thus led to take the equation as the fundamental form of reasoning, and to modify Aristotle's *dictum* in accordance therewith. It may be formulated somewhat as follows—

Whatever is known of a term may be stated of its equal or equivalent.

or in other words,

Whatever is true of a thing is true of its like.

⋮

But the value of the formula must be judged by its results; and I do not hesitate to assert that it not only brings into harmony all the branches of logical doctrine, but that it unites them in close analogy to the corresponding parts of mathematical method. All aspects of mathematical reasoning may, I believe, be considered but as applications of a corresponding axiom of quantity; ...

He has come to the conclusion that this fundamental form is the **Substitution of Equals** as explained via the following picture, where he is discussing the algebra of numbers:

17. ... the wildest possible expression of a process of mathematical inference is shown in the form—

$$\begin{array}{ccccc} a & = & b & & a \\ & & \S & \text{hence} & \S \\ & & c & & c \end{array}$$

where the symbol § means "any conceivable kind of relation between one quantity and another".

... In this all-powerful form we actually seem to have brought together the whole of the processes by which equations are solved, viz. equal addition or subtraction, multiplication or division, involution or evolution, performed upon both sides of the equation at the same time. That most familiar process in mathematical reasoning, of substituting one member of an equation for the other, appears to be the type of all reasoning, and we may fitly name this all-important process the *substitution of equals*.

To express this rule for Jevons' algebra of logic in a slightly more modern form we could use:

$$\frac{\mathrm{A} = \mathrm{B},\ \varphi(\mathrm{B}, \mathrm{C})}{\varphi(\mathrm{A}, \mathrm{C})}$$

where $\varphi(\mathrm{B}, \mathrm{C})$ is any assertion about B and C. As there is no need to refer to C this can be simplified to:

$$\frac{\mathrm{A} = \mathrm{B},\ \varphi(\mathrm{B})}{\varphi(\mathrm{A})}$$

The only properties φ that Jevons considers are *equations*. This is consistent with his view that equations can express any proposition. Furthermore he will always regard $\mathrm{A} = \mathrm{B}$ and $\mathrm{B} = \mathrm{A}$ as interchangeable, and not as a separate rule for equality. Thus it would be appropriate to regard his principle of substitution, that we will call **[Jevons] substitution**, as the following four rules of inference:

$$J1 \quad \frac{\mathrm{A}=\mathrm{B},\ p[\mathrm{B}]=q}{p[\mathrm{A}]=q} \qquad J2 \quad \frac{\mathrm{A}=\mathrm{B},\ p=q[\mathrm{B}]}{p=q[\mathrm{A}]}$$

$$J3 \quad \frac{\mathrm{A}=\mathrm{B},\ p[\mathrm{A}]=q}{p[\mathrm{B}]=q} \qquad J4 \quad \frac{\mathrm{A}=\mathrm{B},\ p=q[\mathrm{A}]}{p=q[\mathrm{B}]}$$

Fig. **45** [Jevons] Substitution

The notation $p[\mathrm{B}]$ is meant to refer to a *single* occurrence of B in the term p.[6]

These rules are very close to the modern rule of **replacement**:[7]

$$\frac{\mathrm{A}=\mathrm{B}}{p[\mathrm{A}]=p[\mathrm{B}]}$$

[Jevons] substitution suffices, as Jevons noted, to derive the *transitivity of equality.* Namely use J1 above with $p[\mathrm{B}]=q$ being the equation $\mathrm{B}=\mathrm{C}$.

From J1 and J3 we can derive the *symmetry of equality.*[8] To do this suppose $\mathrm{A}=\mathrm{B}$ is given. Use J2 with $p=q[\mathrm{B}]$ being $\mathrm{A}=\mathrm{B}$ to derive $\mathrm{A}=\mathrm{A}$. Then use J3 with $p[\mathrm{A}]=q$ being $\mathrm{A}=\mathrm{A}$ to derive $\mathrm{B}=\mathrm{A}$.

For further results we need Jevons' *Law of Identity* which he formulates much later in the article, namely $\mathrm{A}=\mathrm{A}$. For then we can derive $\mathrm{A}+\mathrm{C}=\mathrm{B}+\mathrm{C}$ from $\mathrm{A}=\mathrm{B}$ by using J1 and letting $p[\mathrm{B}]=q$ be the equation $\mathrm{B}+\mathrm{C}=\mathrm{B}+\mathrm{C}$.

It is precisely [Jevons] substitution that he claims is the foundation of correct reasoning.

> **19.** Turning now to apply these considerations to the forms of logical inference, my proposed simplification of the rules of logic is founded upon an obvious extension of the one great process of [Jevons] substitution to all kinds of identity. . . . For it is not difficult to show that all forms of reasoning consist in repeated employment of the universal process of the [Jevons] *substitution of equals*, or, if the phrase be preferred, [Jevons] *substitution of similars.*

In general this is really too strong a claim, but for the equational algebra of logic he is essentially correct if he means that this rule plus suitable axioms suffice. However this is never made precise, except for his discussion of his method of indirect inference:

> **41.** The indirect method is founded upon the law of the [Jevons] substitution of similars as applied with the aid of the fundamental laws of thought. These laws are not to be found in most textbooks of logic, but yet they are necessarily the basis of all reasoning, since they enounce the very nature of similarity or identity. Their existence is assumed or implied, therefore, in the complicated rules of the syllogism, whereas my system is founded upon an immediate application of the laws themselves.

This sounds very promising, but then he follows through with only three laws. It is interesting to note that the reflexive property of equality, that he had ridiculed in **Pure Logic** as useless, is now his first law:

[6] Jevons could not formulate his rule in such a symbolic manner as he did not use symbols for terms. Recall that Boole used φ, f, t, V, etc., to denote terms (he calls them functions), and we use p, q, r, etc., for this purpose. But Jevons only has symbols for classes!

[7] To be precise, the [Jevons] substitution plus the reflexive law for equality is equivalent to the replacement rule plus the reflexive, symmetric and transitive laws of equality.

[8] Jevons never thought it was necessary to treat symmetry as a rule of inference or a law, so of course he did not bother to derive it from [Jevons] substitution.

> The first of these laws, which I have already referred to in an earlier part of this tract (p. 103), is the LAW OF IDENTITY, that *whatever is, is,* or *a thing is identical with itself*; or, in symbols,
>
> $$\mathrm{A} = \mathrm{A}.$$
>
> The second law, THE LAW OF NON-CONTRADICTION, is that *a thing cannot both be and not be*, or that *nothing can combine contradictory attributes*; or, in symbols,
>
> $$\mathrm{A}a = 0,$$
>
> —that is to say, what is both A and not A does not exist, and cannot be conceived.
>
> The third law, that of *excluded middle*, or, as I prefer to call it, the LAW OF DUALITY, asserts the self-evident truth that *a thing either exists or does not exist,* or that *everything either possesses a given attribute or does not possess it.*
>
> Symbolically the law of duality is shown by
>
> $$\mathrm{A} = \mathrm{AB}\cdot|\cdot\mathrm{A}b,$$
>
> in which the sign $\cdot|\cdot$ means alternation, and is equivalent to the true meaning of the disjunctive conjunction *or*. Hence the symbols may be interpreted as, A *is either* B *or not* B.
>
> These laws may seem truisms, and they were ridiculed as such by Locke; but, since they describe the very nature of identity in its three aspects, they must be assumed as true, consciously or unconsciously, and if we can build a system of inference upon them, their self-evidence is surely in our favour.

Note that he has replaced the $+$ of Boole by $\cdot|\cdot$, no doubt to distance his system from ordinary algebra.

In the very first example following this presentation of the three laws he is using a distributive law and commutative law. This system is very clear, but unfortunately very incomplete. There is no explanation as to what happened to the other laws that were formulated in his **Pure Logic** of 1864. The gain in insight regarding the power of [Jevons] substitution is offset by his lack of thoroughness in collecting the needed laws together. This incompleteness may be partly explained by his classification of some laws as "principles of logical symbols":

> **25.** It is desirable at this point to draw attention to the fact that the order in which nouns adjective are stated is a matter of indifference. ... Hence, if A and B represent any two names or terms, their junction as in AB will be taken to indicate anything which unites the qualities of both A and B, and then it follows that
>
> $$\mathrm{AB} = \mathrm{BA}$$
>
> This principle of logical symbols has been fully explained by Dr. Boole in his *Laws of Thought* (pp. 29,30), and also in my *Pure Logic* (p. 15); and its truth will be assumed here without further proof.

He has quietly switched to the use of *extension* for the interpretation of his SYMBOLS in this paper, e.g., one reads:

> **26.** We may now proceed to consider the ordinary proposition of the form
>
> $$\mathrm{A} = \mathrm{AB},$$
>
> which asserts the identity of the class A with a particular part of the class B, ...

Mechanical Aids

As an introduction to his work on mechanical methods of carrying out his indirect inference he says:

> **47.** Objections might be raised against this process of indirect inference, that it is a long and tedious one; and so it is, when thus performed. Tedium indeed is no argument against truth; and if, as I confidently assert, this method gives us the means of solving an infinite number of problems, and arriving at an infinite number of conclusions, which are often demonstrable in no simpler way, and in fact in no other way whatever, no such objections would be of any weight. The fact however is, that almost all the tediousness and liability to mistake may be removed from the process by the use of mechanical aids, which are of several kinds and degrees. While practising myself in the use of the process, I was at once led to the use of the *logical slate*, which consists of a common writing slate, with several series of the combinations of letters engraved upon it . . .

The description that follows is that one has the constituents in 2 through 6 SYMBOLS engraved on the slate. Then he continues:

> **48.** It soon became apparent, however, that if these combinations, instead of being written in fixed order on a slate, were printed upon light movable slips of wood, it would become easy by suitable mechanical arrangements to pick out the combinations in convenient classes, so as immensely to abbreviate the labour of comparison with the premises. This idea was carried out in the *logical abacus*, which I constructed several years ago, and have found useful and successful in the lecture-room for exhibiting the complete solution of logical arguments.

Then he gives an explanation of how this works, and finally turns to his latest mechanical device:

> **53.** In the last paragraph I alluded to a further mechanical contrivance, in which the combination-slips of the abacus should not require to be moved by hand, but could be placed in proper order by the successive pressure of a series of keys or handles. I have since made a successful working model of this contrivance, which may be considered *a machine capable of reasoning*, or of replacing almost entirely the action of the mind in drawing inferences.

Jevons' machine will be discussed in the next section.

Old Logic versus New Logic

This tract contains a substantial discussion on the old logic as compared with the new logic.

> **4.** [Regarding Kant's writings] . . . it might well seem discouraging to logical speculators that he considered logic unimproved in his day since the time of Aristotle, and indeed declared that it could not be improved except in perspicuity. But his opinions have not prevented the improvement of logical doctrine, and are now effectually disproved.
>
> ⋮
>
> **8.** . . . Dr. Boole's remarkable investigations prove that, when once we view the proposition as an equation, all the deductions of the ancient doctrine of logic, and many more, may be arrived at by the processes of algebra. Logic is found to resemble a calculus in which there are only two numbers, 0 and 1, and the analogy of the calculus of quality or fact and the calculus of quantity proves to be perfect.

This passage is misleading. There are other numbers that come into play when one is using Boole's system. For example in the Rule of 0 and 1, it is important to specify that the SYMBOLS are only allowed the two values, but the calculations involved in evaluating the terms can lead to other numbers.

Furthermore, his assessment that the 'analogy' is 'perfect' seems to say that Jevons knows that Boole's 'obscure' mathematical methods are indeed correct, a fact that Jevon's surely did not know to be true. Such positive evaluations would be repeated by later authors, such as Lewis. It seems likely that they simply meant that they were *convinced* that Boole's system gives correct results.

12. ... But it is hardly too much to say that Aristotle committed the greatest and most lamentable of all mistakes in the history of science when he took this kind of proposition as the true type of all propositions, and founded thereon his system. It was by a mere fallacy of accident that he was misled; but the fallacy once committed by a master-mind became so rooted in the minds of all succeeding logicians, by the influence of authority, that twenty centuries have thereby been rendered a blank in the history of logic.
13. ... His syllogism was therefore an edifice in which the corner-stone itself was omitted, and the true system is to be created by suppling this omission, and re-erecting the edifice from the very foundation.

⋮

26. [Regarding A = AB] It may seem when stated in this way to be a truism; but it is not, because it really states in the form of an identity the inclusion of A in a wider class B. Aristotle happened to treat it in the latter aspect only, and the extreme incompleteness of his syllogistic system is due to this circumstance ...

At the end of this paper, on his wonderful discovery of the central role of [Jevons] substitution, we have a case of the writer's blues:

69. I write this tract under the discouraging feeling that the public is little inclined to favour or to inquire into the value of anything of an abstract nature. There are numberless scientific journals and many learned societies, and they readily welcome the minutest details concerning a rare mineral, or an undescribed species, the newest scientific toy, or the latest observations concerning a change in the weather.

⋮

But Logic is under the *ban* of metaphysics. It is falsely supposed to lead to no *useful works*—to be mere speculation; and, accordingly, there is no journal, and no society whatever, devoted to its study. Hardly can a paper on a logical subject be edged into the Proceedings of any learned society except under false pretences.

3. On the Mechanical Performance of Logical Inference (1869)

At this time a **logical machine** that Jevons designed, one that is reminiscent of a very small legless upright piano that could be placed on a table, was constructed, and it displays the consequences of a collection of equations in 4 SYMBOLS after the equations are keyed in.

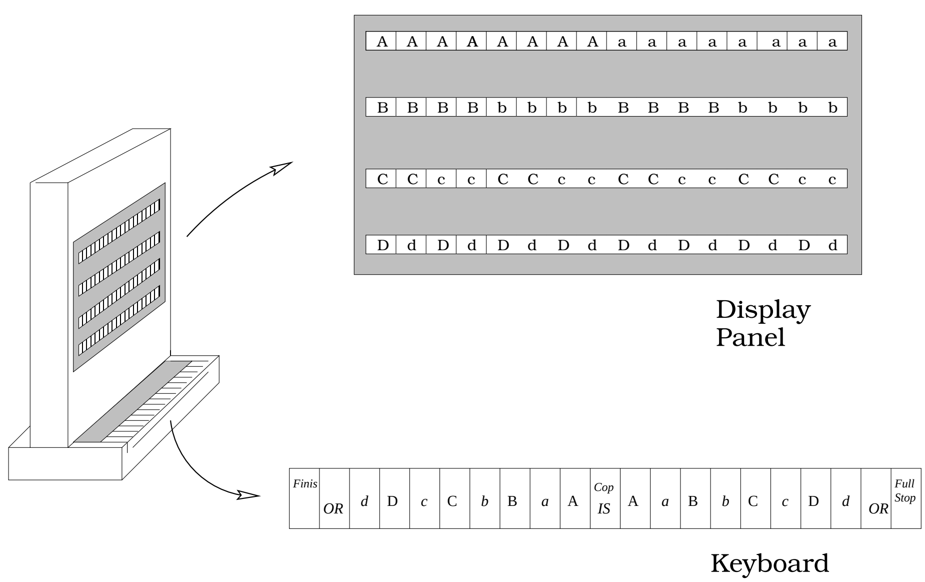

Fig. **46** Jevons' Logical Machine

To key in the equation $\mathrm{A}b{\cdot}|{\cdot}Cd = acD$ one presses the following sequence of keys (reading down the left column, then the center column, and finally the right column):

Left Side	Center	Right Side
Press A	Press *IS*	Press *a*
Press *b*		Press *c*
Press *OR*		Press D
Press C		Press *Full Stop*
Press *d*		

After doing this all the (columns of) constituents that are forced to be 0 by the premise will drop out, leaving blank columns:

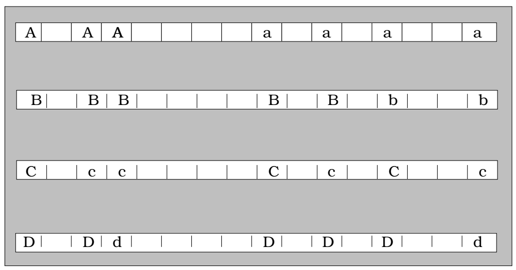

Fig. **47** Results in the Display Panel

Now if one enters another premise further constituents may be forced to drop out. If one enters just an intersection on the left side, say Abc, then just those constituents that include this will be left in the display; pressing *Full Stop* after this, without entering an equation, will return all seven constituents to the display.

4. The Principles of Science (1874)

This 800 page book, subtitled **A Treatise on Logic and Scientific Method**, looks at a wide range of problems and scientific investigations with the idea of illustrating the nature of the reasoning involved in the scientific method by case studies.

We are only focusing on Jevons treatment of deductive logic, a subject that he claims is the prerequisite for inductive logic, and furthermore that logic is prerequisite for the scientific method. His treatment of deductive logic in this book is a fairly leisurely exposition of his earlier work, including various graphical and mechanical aids. The system he gives for modifying Boole's approach to logic combines the best of his **Pure Logic** and **The Substitution of Similars**, and gives us the final version of his system of deductive logic.

The laws are divided into the **three fundamental laws of thought** that are stated informally on page 5 as follows:

> At the base of all thought and science must lie the laws which express the very nature and conditions of the discriminating and identifying powers of mind. These are the so-called Fundamental Laws of Thought, usually stated as follows:—
>
> 1. The Law of Identity. *Whatever is, is.*
> 2. The Law of Contradiction. *A thing cannot both be and not be.*
> 3. The Law of Duality. *A thing must either be or not be.*

and the **special laws which govern the combination of logical terms.**

Collecting together the various laws that he has dispersed over forty pages one has the following table:

Law	Name	Page
Laws of Combination		
A = AA = AAA = &c.	**Law of Simplicity**	33
AB = BA	A **Law of Commutativeness**	35
A·|·A = A	**Law of Unity**	72
A·|·B = B·|·A	A **Law of Commutativeness**	72
A(B·|·C) = AB·|·AC		76
Laws of Thought		
A = A	**Law of Identity**	74
A*a* = ∘	**Law of Contradiction**	74
A = AB·|·A*b*	**Law of Duality**	74

Fig. **48** Jevons' Algebra of Logic 1874

These laws, along with his single rule of inference, [Jevons] substitution, give his deductive logic system for the **Principles of Science** (p. 49):

> By deduction we investigate and unfold the information contained in the premises; and this we can do by one single rule—*For any term occurring in any proposition [Jevons] substitute the term which is asserted in any premise to be identical with it.*

Earlier we showed how to use [Jevons] substitution to derive the **symmetry of equality**, i.e., the rule "if A = B then B = A". However this is not what Jevons did, and instead he says the following (pages 46-47):

> A mathematician would not think it worth while to mention that if $x = y$ then also $y = x$. He would not consider these to be two equations at all, but one equation accidentally written in two different manners.
>
> ⋮
>
> ... so that I shall consider the two forms
>
> $$\mathrm{A} = \mathrm{B} \text{ and } \mathrm{B} = \mathrm{A}$$
>
> to express exactly the same identity differently written.

From the modern point of view Jevons has mixed semantics with syntax here. We consider the axioms and rules of inference to be syntactic entities, i.e., strings of symbols and rules for transforming them. As strings A = B and B = A are certainly distinct, so we can conclude that Jevons did not achieve our basic "strings of symbols" point of view regarding syntax.

The **transitivity of equality** is discussed on page 51, where he shows (while casually changing A = B into B = A) that it is a consequence of [Jevons] substitution. The **associative laws**[9] for both operations are, as usual, implicit in the lack of parentheses in the examples.

[9]In 1880 Peirce attributes the associative laws to Boole and Jevons when, in fact, neither ever mentioned them.

5. Studies in Deductive Logic (1880)

In 1870 Jevons published a primer on traditional deductive and inductive logic called **Elementary Lessons in Logic**, a text that was a stunning success, going through at least 35 reprintings. The text is not only clear but gives the reader a sense of context and insight by discussing the etymology of the special vocabulary of logic.

Ten years later Jevons published **Studies in Deductive Logic**, a book that was primarily an extensive collection of examples and problems, covering traditional logic and his equational version of deductive logic. The presentation of the background is brief, indeed, for his equational approach to logic is described in 3 pages, followed by 75 pages of applications and questions. The system he gives is that of **The Scientific Method**. And, although there is still no symbol routinely used for complementation, and no De Morgan Laws in the above list, he does state that

> . . . every term must have its negative.

And in the preface (pages xv-xvi) he states that some notation of Hugh MacColl, B.A., is:

> . . . very elegant, namely, in using an accent as a sign of negation. A' is the negative of A; and as this accent can be applied with the aid of brackets to terms of any degree of complexity, there may be convenience in using it. Thus $(\mathrm{A}+\mathrm{B})' = \mathrm{A}'\mathrm{B}'; (\mathrm{ABCD}\ldots)' = \mathrm{A}'+\mathrm{B}'+\mathrm{C}'+\mathrm{D}'+\cdots$. I shall take occasionally the liberty of using the accent in this way (see p. 199), but it is not often needed. In the case of single negative terms, I find experimentally that De Morgan's Italic negatives are the best. The Italic *a* is not only far more clearly distinguished from A than is A', but it is written with one pen-stroke less, which in the long run is a matter of importance. The student, of course, can use A' for *a* whenever he finds it convenient.

With this we have the conclusion of Jevons work on logic, a reasonably modern version of the equational logic of Boolean algebra if one includes the associativity laws and the MacColl notation for complements, and one uses [modern] substitution on the laws, and restricts ones attention to the study of ground equational arguments. And indeed, this would be Jevons' last work on logic.

Appendix 1: The Proof System $\mathcal{BR}$

This appendix gives a modern presentation of an equational proof system $\mathcal{BR}$ for **Boolean rings**.[10] The study of Boolean rings did not appear until the mid 1930s, in the work of Marshall Stone, although it is much closer to the work of Boole than the subject of Boolean algebras.

- We have a **set** X **of variables**, whose members we will usually refer to using small latin letters $x, y, z, \ldots$, at the end of the alphabet.
- The **terms** are defined inductively by:
  (i) $\mathbf{0}$ and $\mathbf{1}$ are terms;
  (ii) any variable x is a term;
  (iii) if p is a term then so is $(-p)$;
  (iv) if p and q are terms then so are $(p + q)$ and $(p \cdot q)$.

  The notation $p(x_1, \ldots, x_n)$ means that the variables that appear in p are *among* the variables in the list $x_1, \ldots, x_n$.
- An **equation** is an expression of the form $p = q$ where p and q are terms.
- The **Axioms** are

1.	$x + y \;=\; y + x$	commutative law
2.	$x \cdot y \;=\; y \cdot x$	commutative law
3.	$x + (y + z) \;=\; (x + y) + z$	associative law
4.	$x \cdot (y \cdot z) \;=\; (x \cdot y) \cdot z$	associative law
5.	$x + \mathbf{0} \;=\; x$	additive zero
6.	$x \cdot \mathbf{1} \;=\; x$	multiplicative unit
7.	$x + (-x) \;=\; \mathbf{0}$	additive inverse
8.	$x \cdot x \;=\; x$	idempotent multiplication
9.	$x \cdot (y + z) \;=\; (x \cdot y) + (x \cdot z)$	distributive law

[10] No proofs will be given in this appendix as the results follow easily from standard references such as the detailed introduction to the equational theory of Boolean rings in Burris and Sankappanavar [**3**], Chapter IV.

- The **Rules of Inference** are as follows,[11] where the letters p, q, r, s_i refer to arbitrary but fixed terms:

$$\left.\frac{}{p \;=\; p}\right\} \quad \text{reflexive rule}$$

$$\left.\frac{p \;=\; q}{q \;=\; p}\right\} \quad \text{symmetric rule}$$

$$\left.\frac{p \;=\; q,\; q \;=\; r}{p \;=\; r}\right\} \quad \text{transitive rule}$$

$$\left.\frac{p(x_1, \ldots, x_n) \;=\; q(x_1, \ldots, x_n)}{p(s_1, \ldots, s_n) \;=\; q(s_1, \ldots, s_n)}\right\} \quad \text{[modern] substitution rule}$$

$$\left.\frac{p \;=\; q}{s[p] \;=\; s[q]}\right\} \quad \text{replacement rule}$$

Given a set S of equations we say that an equation $p = q$ **has a derivation from** S in this proof system if there is a sequence of equations

$$p_1 \;=\; q_1,\; \cdots,\; p_k \;=\; q_k$$

such that

- each equation in the sequence is either
  (i) an axiom, or
  (ii) it is in S, or
  (iii) it is the result of applying a rule of inference to previous equations in the sequence,
- and the equation $p = q$ is $p_k = q_k$.

If one can derive $p = q$ from S in $\mathcal{BR}$ we write $S \vdash_{\mathcal{BR}} p = q$, or simply say that **the argument** $S \;\therefore p = q$ **has a derivation** in this system.

Two equations $p = q$ and $r = s$ are **$\mathcal{BR}$-equivalent** if

$$p = q \;\vdash_{\mathcal{BR}}\; r = s \qquad \text{and} \qquad r = s \;\vdash_{\mathcal{BR}}\; p = q.$$

We say two collections of equations S_1 and S_2 are $\mathcal{BR}$-equivalent if

$$\begin{array}{lll} S_1 \;\vdash_{\mathcal{BR}}\; p = q & \text{for } p = q \in S_2, & \text{and} \\ S_2 \;\vdash_{\mathcal{BR}}\; p = q & \text{for } p = q \in S_1. & \end{array}$$

Every equation $p = q$ is $\mathcal{BR}$-equivalent to an equation in the form $s = \mathbf{0}$, namely $p = q$ is $\mathcal{BR}$-equivalent to $p - q = \mathbf{0}$. Thus every equational argument can be put in the form

$$p_1 = \mathbf{0}, \ldots, p_k = \mathbf{0} \qquad \therefore p = \mathbf{0}$$

as far as $\vdash_{\mathcal{BR}}$ is concerned.

Abbreviating Terms

[11]Instead of the version of the replacement rule used here one can use the following three simpler rules:

$$\frac{p = q}{-p = -q} \qquad \frac{p = q}{p + r = q + r} \qquad \frac{p = q}{p \cdot r = q \cdot r}$$

Now that we have carefully defined the terms of $\mathcal{BR}$ it is important to inform the reader that we like to use abbreviated and often ambiguous notation for terms. Here are the basic conventions:

1. Omit outer parentheses, e.g., $x + (y + z)$ instead of $(x + (y + z))$.
2. Write xy instead of $x \cdot y$.
3. Specify that $-$ has greater binding power than $\cdot$, which in turn has greater binding power than $+$. Thus $x + y \cdot z$ means $x + (y \cdot z)$ and not $(x + y) \cdot z$.
4. Omit parentheses when several terms are added (or multiplied) together, e.g., $x + y + z + w$ instead of $x + ((y + z) + w)$, and $xyzw$ instead of $(x(yz))w$. These expressions are ambiguous as one has several ways to insert parentheses to create a term. However, any two terms p and q with the same abbreviated form are *$\mathcal{BR}$-equivalent*, i.e., $\vdash_{\mathcal{BR}} p = q$.
5. $\sum_{i=1}^{k} p_i$ is shorthand for $p_1 + \cdots + p_k$, and $\prod_{i=1}^{k} p_i$ is shorthand for $p_1 \cdots p_k$.
6. For n a positive integer np is shorthand for a sum of n p's, and p^n is shorthand for a product of n p's.

Using these conventions usually helps with the communication of results.

The Boolean Ring of Subclasses of U

There is a natural way to associate Boolean rings with the logic of classes. For let U be a given universe of discourse. If X and Y are subclasses of U let $X + Y$ be the **symmetric difference** of X and Y, that is, the class of elements in X or in Y but not in both. Let $X \cdot Y$ be the **intersection** of X and Y, and let $-X$ just be X. And let $\mathbf{0}$ be **the empty class** and **1 the class** U. The subclasses of U with these operations is an example of a **model** of the axioms of Boolean rings, i.e., the axioms of Boolean rings are true in this interpretation.

One can use the proof system $\mathcal{BR}$ to study this model. For example from the idempotent law we have $(x + x)^2 = x + x$, and then one can derive the law $x + x = \mathbf{0}$. Interpreting this we have: the symmetric difference of any class with itself is empty.

Adapting $\mathcal{BR}$ to Boole's Algebra of Logic

To adapt the proof system $\mathcal{BR}$ to the algebra of logic of the 19th century one could construct an equational system without variables, as we will do in Appendix 3, but we prefer to stay as close as possible to the given system $\mathcal{BR}$. We will simply augment $\mathcal{BR}$ with a set C of constants to make the proof system $\mathcal{BR}[C]$. The only change we need to make is to add the constant symbols from C to the terms. Then the work of Boole can be slightly modified to focus on determining when a ground equational argument

$$p_1(\vec{a}, \vec{b}\,) \;=\; q_1(\vec{a}, \vec{b}\,),\; \ldots\;,\; p_k(\vec{a}, \vec{b}\,) \;=\; q_k(\vec{a}, \vec{b}\,) \qquad \therefore\; p(\vec{b}\,) \;=\; q(\vec{b}\,)$$

has a derivation in this system.

There is a local **soundness** and **completeness** theorem which is as follows. Given any U that is not empty, an argument

$$p_1(\vec{a}\,) = q_1(\vec{a}\,), \ldots , p_k(\vec{a}\,) = q_k(\vec{a}\,) \qquad \therefore\; p(\vec{a}\,) = q(\vec{a})$$

has a derivation in $\mathcal{BR}[C]$ iff for any subclasses $A_1, \ldots , A_n$ of U for which the equations

$$p_1(\vec{A}\,) = q_1(\vec{A}\,), \ldots , p_k(\vec{A}\,) = q_k(\vec{A}\,)$$

hold we also have $p(\vec{A}\,) = q(\vec{A}\,)$ holding.

Here are the main theorems of Boole, adapted to this system, where $\mathbf{Z}_2$ is the 2-element Boolean ring given by:

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

	$-$
0	0
1	1

The Rule of 0 and 1 for $\mathcal{BR}$

A ground equational argument

$$p_1(\vec{a}\,) = q_1(\vec{a}\,), \ldots , p_k(\vec{a}\,) = q_k(\vec{a}\,) \qquad \therefore\ p(\vec{a}\,) = q(\vec{a}\,)$$

has a derivation in the equational proof system $\mathcal{BR}[C]$ iff every 0–1 assignment of C that makes the premises true in $\mathbf{Z}_2$ also makes the conclusion true in $\mathbf{Z}_2$.

In the Introduction we applied Boole's Rule of 0 and 1 to the commutative law. Now let us apply the Rule of 0 and 1 for Boolean rings:

a	b	$a+b$	$b+a$
1	1	0	0
1	0	1	1
0	1	1	1
0	0	0	0

Fig. **49** Applying the Rule of 0 and 1

Due to the fact that $\mathbf{Z}_2$ is the only (nontrivial) subdirectly irreducible Boolean ring one actually has an **Extended Rule of 0 and 1** that applies to all equational arguments, not just ground equational arguments. For example one can use it to justify the *quantified* version $x + y = y + x$ of the commutative law as well as the argument $ax = bx\ \therefore a = b$.[12] Of course a similar argument using only ground equations, $ac = bc\ \therefore a = b$, is not valid.

Given a list $a_1, \ldots , a_n$ of constants from C we define an **$\vec{a}$-constituent** to be a term of the form

$$\widehat{a}_1 \cdots \widehat{a}_n$$

where each $\widehat{a}_i$ is either a_i or $1 - a_i$.

Then if $t_1, \ldots , t_{2^n}$ are the 2^n distinct $\vec{a}$-constituents we have

$$\begin{array}{lll} \vdash_{\mathcal{BR}}\ t_i \wedge t_i \ = \ t_i & & \text{for all } i \\ \vdash_{\mathcal{BR}}\ t_i \wedge t_j \ = \ \mathbf{0} & & \text{for } i \neq j, \text{ and} \\ \vdash_{\mathcal{BR}}\ \mathbf{1} \ = \ t_1 \vee \cdots \vee t_{2^n}. & & \end{array}$$

Expansion Theorem

Any term $p(\vec{a}, \vec{b}\,)$ has an expansion on the symbols $\vec{a}$ given by

$$p(\vec{a}, \vec{b}\,) \ = \ p(\mathbf{1}, \ldots , \mathbf{1}, \vec{b}\,)a_1 \cdots a_n \ + \ \cdots \ + \ p(\mathbf{0}, \ldots , \mathbf{0}, \vec{b}\,)(\mathbf{1} - a_1) \cdots (\mathbf{1} - a_n).$$

Reduction Theorem

[12] Such arguments, using a mixture of quantified variables and unquantified constants, were not used by Boole or Jevons.

A system of equations

$$\begin{cases} p_1 &= \mathbf{0} \\ & \vdots \\ p_k &= \mathbf{0} \end{cases}$$

is $\mathcal{BR}$-equivalent to the single equation obtained by adding all the elementary functions of $p_1, \ldots, p_k$ together and setting the sum equal to $\mathbf{0}$, i.e.,

$$(p_1 + \cdots + p_k) \ + \ \left(\sum_{i<j} p_i p_j\right) \ + \cdots + \ (p_1 \cdots p_k) \ = \ \mathbf{0}.$$

Elimination Theorem

The most general equation that one can deduce from

$$p(\vec{a}, \vec{b}\,) \ = \ \mathbf{0}$$

that involves only the symbols $\vec{b}$ from C is

$$q(\vec{b}\,) \ = \ \mathbf{0},$$

where

$$q(\vec{b}\,) \ = \ p(\mathbf{1}, \ldots, \mathbf{1}, \vec{b}\,) \ \cdots \ p(\mathbf{0}, \ldots, \mathbf{0}, \vec{b}\,).$$

Appendix 2: The Proof System $\mathcal{BA}$

This appendix gives a modern presentation of an equational proof system $\mathcal{BA}$ for **Boolean algebra**.[13]

- We have a **set** X **of variables**, whose members we will usually refer to using small latin letters $x, y, z, \ldots$, at the end of the alphabet.
- The **terms** are defined inductively by:
  (i) $\mathbf{0}$ and $\mathbf{1}$ are terms;
  (ii) any variable x is a term;
  (iii) if p is a term then so is (p');
  (iv) if p and q are terms then so are $(p \vee q)$ and $(p \wedge q)$.
  The notation $p(x_1, \ldots, x_n)$ means that the variables that appear in p are *among* the variables in the list $x_1, \ldots, x_n$.
- An **equation** is an expression of the form $p = q$ where p and q are terms.
- The **Axioms** are

1.	$x \vee x = x$	idempotent law
2.	$x \wedge x = x$	idempotent law
3.	$x \vee y = y \vee x$	commutative law
4.	$x \wedge y = y \wedge x$	commutative law
5.	$x \vee (y \vee z) = (x \vee y) \vee z$	associative law
6.	$x \wedge (y \wedge z) = (x \wedge y) \wedge z$	associative law
7.	$x \vee (x \wedge y) = x$	absorption law
8.	$x \wedge (x \vee y) = x$	absorption law
9.	$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$	distributive law
10.	$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$	distributive law
11.	$(x \vee y)' = x' \wedge y'$	De Morgan law
12.	$(x \wedge y)' = x' \vee y'$	De Morgan law
13.	$x \vee x' = \mathbf{1}$	
14.	$x \wedge x' = \mathbf{0}$	

[13]No proofs will be given in this appendix as the results follow easily from standard references such as the detailed introduction to the equational theory of Boolean algebras in Burris and Sankappanavar [**3**], Chapter IV.

- The **Rules of Inference** are as follows,[14] where the letters p, q, r, s_i refer to arbitrary but fixed terms:

$$\left.\frac{}{p \;=\; p}\right\}\quad \text{reflexive rule}$$

$$\left.\frac{p \;=\; q}{q \;=\; p}\right\}\quad \text{symmetric rule}$$

$$\left.\frac{p \;=\; q,\; q \;=\; r}{p \;=\; r}\right\}\quad \text{transitive rule}$$

$$\left.\frac{p(x_1,\dots,x_n) \;=\; q(x_1,\dots,x_n)}{p(s_1,\dots,s_n) \;=\; q(s_1,\dots,s_n)}\right\}\quad \text{[modern] substitution rule}$$

$$\left.\frac{p \;=\; q}{s[p] \;=\; s[q]}\right\}\quad \text{replacement rule}$$

Given a set S of equations we say that an equation $p = q$ **has a derivation from** S in this proof system if there is a sequence of equations

$$p_1 \;=\; q_1,\; \cdots,\; p_k \;=\; q_k$$

such that

- each equation in the sequence is either
  (i) an axiom, or
  (ii) it is in S, or
  (iii) it is the result of applying a rule of inference to previous equations in the sequence,
- and the equation $p = q$ is $p_k = q_k$.

If one can derive $p = q$ from S in $\mathcal{BA}$ we write $S \vdash_{\mathcal{BA}} p = q$, or simply say that **the argument** $S\ \therefore p = q$ **has a derivation** in this system.

Two equations $p = q$ and $r = s$ are **$\mathcal{BA}$-equivalent** if

$$p = q \vdash_{\mathcal{BA}} r = s \qquad \text{and} \qquad r = s \vdash_{\mathcal{BA}} p = q.$$

We say two collections of equations S_1 and S_2 are $\mathcal{BA}$-equivalent if

$$\begin{aligned} S_1 &\vdash_{\mathcal{BA}} p = q \qquad \text{for } p = q \in S_2 \qquad \text{and} \\ S_2 &\vdash_{\mathcal{BA}} p = q \qquad \text{for } p = q \in S_1. \end{aligned}$$

Every equation $p = q$ is $\mathcal{BA}$-equivalent to an equation in the form $s = \mathbf{0}$, namely $p = q$ is $\mathcal{BA}$-equivalent to $(p \wedge q') \vee (p' \wedge q) = \mathbf{0}$. Thus every equational argument can be put in the form

$$p_1 = \mathbf{0}, \dots, p_k = \mathbf{0} \qquad \therefore p = \mathbf{0}$$

as far as $\vdash_{\mathcal{BA}}$ is concerned.

We will adopt **abbreviation conventions** much like those described in Appendix 1, except that we do not assign a binding priority to the two binary operations $\vee$ and $\wedge$.

[14]Instead of the version of the replacement rule used here one can use the following three simpler rules:

$$\frac{p = q}{p' = q'} \qquad \frac{p = q}{p \vee r = q \vee r} \qquad \frac{p = q}{p \wedge r = q \wedge r}$$

The Boolean Algebra of Subclasses of U

There is a natural way to associate Boolean algebra with the logic of classes. For let U be a given universe of discourse. If X and Y are subclasses of U let $X \vee Y$ be the **union** of X and Y, that is, the class of elements in X or in Y or in both. Let $X \wedge Y$ be the **intersection** of X and Y, and let X' be **the complement** of X. And let **0** be **the empty class** and **1 the class** U. The subclasses of U with these operations provide a **model** of the axioms of Boolean algebra.

Adapting $\mathcal{BA}$ to Jevons' Algebra of Logic

To adapt the proof system $\mathcal{BA}$ to the algebra of logic of the 19th century we will simply augment $\mathcal{BA}$ with a set C of constants to make the proof system $\mathcal{BA}[C]$. The only change we need to make is to add the constant symbols from C to the terms. Then the work of Jevons can be viewed as determining when a ground equational argument

$$p_1(\vec{a},\vec{b}\,) \;=\; q_1(\vec{a},\vec{b}\,),\;\ldots\;,\; p_k(\vec{a},\vec{b}\,) \;=\; q_k(\vec{a},\vec{b}\,) \qquad \therefore\; p(\vec{b}\,) \;=\; q(\vec{b}\,)$$

has a derivation in this system. One can translate universal categorical propositions into equations as Jevons did, once we convert Jevons' notation to modern notation, namely

translate A + B as $A \vee B$
translate A·|·B as $A \vee B$
translate AB as $A \wedge B$
translate a as A'.

There is a local **soundness** and **completeness** theorem which is as follows. Given any U that is not empty, an argument

$$p_1(\vec{a}\,) = q_1(\vec{a}\,), \ldots , p_k(\vec{a}\,) = q_k(\vec{a}\,) \qquad \therefore\; p(\vec{a}\,) = q(\vec{a}\,)$$

has a derivation in $\mathcal{BA}[C]$ iff for any subclasses $A_1, \ldots , A_n$ of U for which the equations

$$p_1(\vec{A}\,) = q_1(\vec{A}\,), \ldots , p_k(\vec{A}\,) = q_k(\vec{A}\,)$$

hold we also have $p(\vec{A}) = q(\vec{A})$ holding.

Here are the main theorems of Boole, adapted to this system,[15] where $\mathbf{B}_2$ is the 2-element Boolean algebra given by:

$\vee$	0	1
0	0	1
1	1	1

$\wedge$	0	1
0	0	0
1	0	1

	$'$
0	1
1	0

The Rule of 0 and 1 for $\mathcal{BA}$

A ground equational argument

$$p_1(\vec{a}\,) = q_1(\vec{a}\,), \ldots , p_k(\vec{a}\,) = q_k(\vec{a}\,) \qquad \therefore\; p(\vec{a}\,) = q(\vec{a})$$

has a derivation in the proof system $\mathcal{BA}[C]$ iff every 0–1 assignment of C that makes the premises true in $\mathbf{B}_2$ also makes the conclusion true in $\mathbf{B}_2$.

[15] None of these were stated by Jevons. One can find them, except for the Rule of 0 and 1, in Schröder's Algebra der Logik in the 1890s. Schröder says that the formulation of the Elimination Theorem (for Boolean algebra) is due to him.

Due to the fact that $\mathbf{B}_2$ is the only (nontrivial) subdirectly irreducible Boolean algebra one actually has an **Extended Rule of 0 and 1** that applies to all equational arguments, not just ground equational arguments.

Given a list $a_1, \ldots, a_n$ of constants from C we define an **$\vec{a}$-constituent** to be a term of the form

$$\widehat{a}_1 \wedge \cdots \wedge \widehat{a}_n$$

where each $\widehat{a}_i$ is either a_i or a_i'.

Thus if $t_1, \ldots, t_{2^n}$ are the 2^n distinct $\vec{a}$-constituents we have

$$\begin{array}{lll} \vdash_{\mathcal{BA}} & t_i \wedge t_i \ = \ t_i & \text{for all } i \\ \vdash_{\mathcal{BA}} & t_i \wedge t_j \ = \ \mathbf{0} & \text{for } i \neq j, \text{ and} \\ \vdash_{\mathcal{BA}} & \mathbf{1} \ = \ t_1 \vee \cdots \vee t_{2^n}. & \end{array}$$

Expansion Theorem

Any term $p(\vec{a}, \vec{b}\,)$ has an expansion on the symbols $\vec{a}$ given by

$$p(\vec{a}, \vec{b}\,) \ = \ \big[p(\mathbf{1}, \ldots, \mathbf{1}, \vec{b}\,) \wedge a_1 \wedge \cdots \wedge a_n\big] \ \vee \ \cdots \ \vee \ \big[p(\mathbf{0}, \ldots, \mathbf{0}, \vec{b}\,) \wedge a_1' \wedge \cdots \wedge a_n'\big].$$

Reduction Theorem

A system of equations

$$\begin{cases} p_1 & = \ \mathbf{0} \\ & \vdots \\ p_k & = \ \mathbf{0} \end{cases}$$

is $\mathcal{BA}$-equivalent to the single equation

$$p_1 \vee \cdots \vee p_k \ = \ \mathbf{0}.$$

Elimination Theorem

The most general equation that one can deduce from

$$p(\vec{a}, \vec{b}\,) \ = \ \mathbf{0}$$

that involves only the symbols $\vec{b}$ from C is

$$q(\vec{b}\,) \ = \ \mathbf{0},$$

where

$$q(\vec{b}\,) \ = \ p(\mathbf{1}, \ldots, \mathbf{1}, \vec{b}\,) \wedge \ \cdots \ \wedge p(\mathbf{0}, \ldots, \mathbf{0}, \vec{b}\,).$$

Appendix 3: The Proof System $\mathcal{AB}$

In Chapter 5 we noted that Jevons praised the perfect analogy between Boole's algebra of logic and a restricted algebra of numbers, a fact that Jevons in all likelihood did not know to be true. In C.L. Lewis's classic text of 1918, A Survey of Symbolic Logic, one finds (page 55) his assessment of Boole's methods:

> Nevertheless, this apparently arbitrary way of using uninterpretable algebraic processes is thoroughly sound.

It also seems unlikely that Lewis knew this, except for some examples. Boole asserts the completeness of his system in his 1847 book, with no justification, and then expands this system in his 1854 book. He no longer claims it is complete, but instead takes a totally new guiding principle, the Rule of 0 and 1, which asserts a perfect correspondence between two systems. This is indeed correct, but again there is absolutely no justification given. This gap was not filled until 1976, when Hailperin showed how to construct secure foundations for Boole's methods (although we find that Hailperin's treatment needs some clarification). We do not know of any such scholarly evaluation of Boole's work that was available before Hailperin's book.

This appendix gives a presentation of the algebra of Boole that is both faithful to the books written by Boole and is easily understandable to those with experience in modern algebra. Our presentation is quite close to that of Hailperin in **Boole's Logic and Probability**, but differs in the treatment of the law of substitution and the detail given to the rules of inference.

First (and foremost) we will treat the part of the algebra of Boole that does not involve division ($\div$). Hailperin proposes a direct power $\mathbf{Z}^U$ of the ring of integers $\mathbf{Z}$ as a model of this part of Boole's logic. Here U is the universe (but not Boole's universe of "everything"). Recall that $\mathbf{Z}^U$ is the ring $(Z^U, +, \cdot, -, \mathbf{1}, \mathbf{0})$ where Z^U is the collection of all functions $f : U \to Z$ with $\mathbf{0}$ being the constant function $\mathbf{0}(u) = 0$, and $\mathbf{1}$ being the constant function $\mathbf{1}(u) = 1$. And the operations are defined coordinatewise:

$$\begin{aligned}(-f)(u) &= -(f(u))\\ (f+g)(u) &= f(u)+g(u)\\ (f\cdot g)(u) &= f(u)\cdot g(u)\end{aligned}$$

The idempotents of $\mathbf{Z}^U$ are the characteristic functions $\mathcal{X}_A$ of subclasses A of U, and thus one has a natural identification of idempotents with subclasses of U.

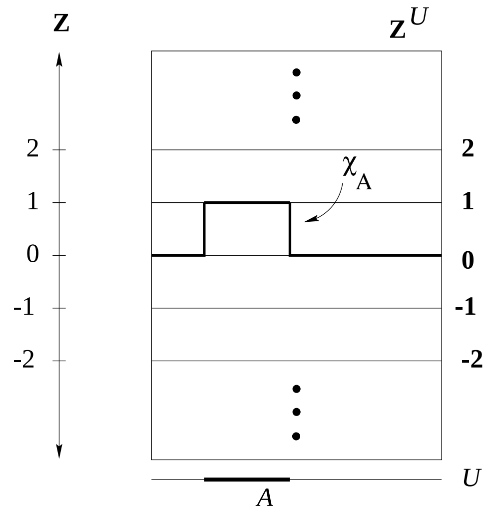

Fig. **50** Visualizing Hailperin's Model of Signed Multisets

Hailperin calls an element h of $\mathbf{Z}^U$ a **signed multiset**, and one can think of it as a generalized class in the sense that each element $u \in U$ is designated to be appearing $h(u)$ times in h. The genuine classes correspond to those h for which each element u appears 0 or 1 times, namely the idempotents.

From the following facts about the idempotents in $\mathbf{Z}^U$:

- $\mathcal{X}_\emptyset \ = \ \mathbf{0}$
- $\mathcal{X}_U \ = \ \mathbf{1}$
- $\mathcal{X}_A \cdot \mathcal{X}_B \ = \ \mathcal{X}_{A\cap B}$
- $\mathcal{X}_A + \mathcal{X}_B \ = \ \mathcal{X}_{A\cup B} + \mathcal{X}_{A\cap B}$
- $\mathcal{X}_A - \mathcal{X}_B \ = \ \mathcal{X}_{A\setminus B} - \mathcal{X}_{B\setminus A}$

we can conclude that:

- The $\mathbf{0}$ element of $\mathbf{Z}^U$ corresponds to the empty subclass $\emptyset$ of U.
- The $\mathbf{1}$ element of $\mathbf{Z}^U$ corresponds to the universe U.
- The multiplication $\mathcal{X}_A \cdot \mathcal{X}_B$ of idempotents corresponds to the intersection $A \cap B$.
- The sum $\mathcal{X}_A + \mathcal{X}_B$ of two idempotents corresponds to a class (i.e., is idempotent) iff A and B are disjoint classes, and in this case the sum corresponds to the union $A \cup B$.
- The difference $\mathcal{X}_A - \mathcal{X}_B$ of two idempotents corresponds to a class iff B is a subclass of A, and in this case the difference corresponds to the class difference $A \setminus B$.

Using the correspondence with the idempotents we can think of the ring $\mathbf{Z}^U$ as an extension of the the collection of subclasses of U, with operations $+, \cdot, -$ that are completely defined, and that satisfy the basic laws formulated by Boole (as $\mathbf{Z}^U$ is a ring). Furthermore, taking the notion of interpretable to mean idempotent, the ring $\mathbf{Z}^U$ exhibits the basic interpretability behavior that Boole noted.

We want to describe a proof system for the equational theory of $\mathbf{Z}^U$ that corresponds to Boole's axioms and rules of inference, and prove that it has the desired properties. Hailperin says ([**6**] p. 140) that the algebra of signed multisets has the following axioms:[16]

---

[16]This is described as **the theory of commutative rings with unit and with no non-zero nilpotents, either multiplicative or additive**.

H1.	$A + B = B + A$
H2.	$A + (B + C) = (A + B) + C$
H3.	$A + 0 = A$
H4.	$A + X = 0$ has a (unique) solution for X
H5.	$AB = BA$
H6.	$A(BC) = (AB)C$
H7.	$A1 = A$
H8.	$A(B + C) = AB + AC$
H9.	$1 \neq 0$
H10.	$A^2 = 0$ only if $A = 0$
H11$_n$.	$nA = 0$ only if $A = 0$ $(n = 1, 2, 3, \ldots)$

Fig. **51** Hailperin's Axioms for Signed Multisets

One important feature that is missing from Hailperin's description is a formulation of the *rules of inference*,[17] even though Boole made a clear attempt to describe the principles underlying his proof system. In the following we give a proof system $\mathcal{AB}$ for working with equations that is based on Hailperin's signed multisets, but it differs in three respects. First there is no substitution rule as there are only constants (and no variables) for the idempotents. Secondly the rules of inference are completely specified. Thirdly, we do not need H10.

As explained in the Introduction, the literal symbols x, y, ... in Boole's logic behave, for the most part, like *symbols for constants*. To reinforce this point of view we change his symbols x, y, ... to the symbols that we like to use for constants, such as a, b, ... We will set up an **equational logic *without variables***.

1. The system $\mathcal{AB}$

We use the function and constant symbols of **the language of rings**: $+, \cdot, -, \mathbf{0}, \mathbf{1}$, plus a set C of constant symbols as names for classes. We use the boldface $\mathbf{0}$ and $\mathbf{1}$ to distinguish the constant symbols from the integers 0 and 1. The **terms** for the language are defined inductively by:

- The constants $\mathbf{0}$ and $\mathbf{1}$ are terms.
- The constant symbols in C are terms.
- If p and q are terms then so are $(-p)$, $(p + q)$, $(p \cdot q)$.

These terms are clearly *ground* terms as they have no variables. An **equation** is an expression of the form $p = q$ where p and q are terms. Equations are clearly *ground* equations. The following are

[17]There is some suggestion that Hailperin is interested in full first-order logic as he says ([**6**], p. 140):

> Unlike the (elementary) theory of Boolean algebras, the theory of SM algebras ... is undecidable.

This result certainly applies to the full first-order theory of signed multisets. However if one restricts oneself to the kinds of formulas that express the arguments that occur in Boole's text, namely a conjunction of equations implies an equation (to express the assertion that the premises imply the conclusion) then such formulas are of course *decidable* (by the Rule of 0 and 1).

usual abbreviations, where n is a positive integer and p is a term:

$$\mathbf{n} \quad \text{for} \quad \underbrace{\mathbf{1} + \cdots + \mathbf{1}}_{n \text{ times}} \qquad -\mathbf{n} \quad \text{for} \quad -(\underbrace{\mathbf{1} + \cdots + \mathbf{1}}_{n \text{ times}})$$

$$np \quad \text{for} \quad \underbrace{p + \cdots + p}_{n \text{ times}} \qquad (-n)p \quad \text{for} \quad -(\underbrace{p + \cdots + p}_{n \text{ times}})$$

$0p = \mathbf{0}$, where p is a term.

We adopt the conventions for abbreviating terms described in Appendix 1.

Here is the system $\mathcal{AB}$ (*the Algebra of Boole!*):

Axioms

- For each constant symbol a in C we have the axiom $a^2 = a$.
- The laws of "Common Algebra", where p, q, r represent any ground terms:

$$\begin{array}{ll} p + \mathbf{0} = p & p\mathbf{1} = p \\ p + (-p) = \mathbf{0} & pq = qp \\ p + (q + r) = (p + q) + r & p(qr) = (pq)r \\ p + q = q + p & \end{array}$$

$$p(q + r) = pq + pr$$

Rules of Inference

Equality is reflexive, symmetric and transitive

$$\frac{}{p = p} \qquad \frac{p = q}{q = p} \qquad \frac{p = q,\ q = r}{p = r}$$

Replacement

$$\frac{p = q}{-p = -q} \qquad \frac{p = q}{p + r = q + r} \qquad \frac{p = q}{pr = qr}$$

Additively nonnilpotent

$$\frac{np = \mathbf{0}}{p = \mathbf{0}} \qquad \text{for } n \in \{1, 2, \dots\}$$

Fig. **52** The Proof System $\mathcal{AB}$

This is not a usual equational logic because of the Additively Nonnilpotent rule of inference. The models of $\mathcal{AB}$ are closed under subalgebras and direct products, but not under quotients.

Given a set S of equations and an equation $p = q$, a **derivation** of $p = q$ from S in the system $\mathcal{AB}$ is a finite sequence of equations $p_1 = q_1, \dots, p_k = q_k$ such that each $p_i = q_i$ is either an axiom, or is in S, or can be obtained from previous equations by applying an inference rule, and $p_k = q_k$ is the equation $p = q$. If such a derivation exists we write $S \vdash_{\mathcal{AB}} p = q$, or say that **the argument** $S \ \therefore\ p = q$ **has a derivation in** $\mathcal{AB}$. If $S = \emptyset$ we just write $\vdash_{\mathcal{AB}} p = q$.

Two equations $p = q$ and $r = s$ are $\mathcal{AB}$**-equivalent** if

$$p = q \ \vdash_{\mathcal{AB}} \ r = s \qquad \text{and} \qquad r = s \ \vdash_{\mathcal{AB}} \ p = q.$$

We say two collections of equations S_1 and S_2 are $\mathcal{AB}$-equivalent if

$$\begin{array}{lll} S_1 \ \vdash_{\mathcal{AB}} \ p = q & \text{for } p = q \in S_2 & \text{and} \\ S_2 \ \vdash_{\mathcal{AB}} \ p = q & \text{for } p = q \in S_1. & \end{array}$$

Every equation $p = q$ is $\mathcal{AB}$-equivalent to an equation in the form $s = \mathbf{0}$, namely $p = q$ is $\mathcal{AB}$-equivalent to $p - q = \mathbf{0}$. Thus every equational argument for $\mathcal{AB}$ can be put in the form

$$p_1 = \mathbf{0}, \dots, p_k = \mathbf{0} \ \vdash_{\mathcal{AB}} \ p = \mathbf{0}.$$

2. The Rule of 0 and 1

Now we will present an elementary proof, one that only requires working with the integers, that the proof system $\vdash_{\mathcal{AB}}$ is indeed captured by the Rule of 0 and 1. Perhaps the logicians of the 19th century and the early part of the 20th century grasped the essence of why this was true, but they gave no indication of this in their writings.

Let $\mathcal{I}_{\mathbf{Z}}$ be the collection of 0–1 **interpretations** of C in $\mathbf{Z}$, that is, the collection of mappings $\alpha : C \to \{0, 1\}$. Any such α can be naturally extended to all terms by the inductive definition:

$$\begin{aligned} \alpha\mathbf{0} &= 0 \\ \alpha\mathbf{1} &= 1 \\ \alpha(-p) &= -\alpha p \\ \alpha(p+q) &= \alpha p + \alpha q \\ \alpha(p \cdot q) &= \alpha p \cdot \alpha q \end{aligned}$$

For $\alpha \in \mathcal{I}_{\mathbf{Z}}$ we say $(\mathbf{Z}, \alpha)$ **satisfies** $p = q$, written $(\mathbf{Z}, \alpha) \models p = q$, if $\alpha p = \alpha q$, i.e., the equation $p = q$ is true in the integers after assigning the constants a the values αa. For example if $\alpha a = 6$ and $\alpha b = 3$ we have $(\mathbf{Z}, \alpha) \models a = b + b$.

And for S a collection of equations we say $(\mathbf{Z}, \alpha)$ **satisfies** S, written $(\mathbf{Z}, \alpha) \models S$, if $(\mathbf{Z}, \alpha) \models p = q$ for each equation $p = q$ in S. We write

$$S \models_{\mathbf{Z}} p = q$$

if, for every $\alpha \in \mathcal{I}_{\mathbf{Z}}$, we have

$$(\mathbf{Z}, \alpha) \models S \qquad \text{implies} \qquad (\mathbf{Z}, \alpha) \models p = q,$$

i.e., for any $\alpha \in \mathcal{I}_{\mathbf{Z}}$, if all the equations in S are true in $\mathbf{Z}$ under the interpretation α then the equation $p = q$ is also true in $\mathbf{Z}$ under this interpretation.

Two equations $p = q$ and $r = s$ are **Z-equivalent** if

$$p = q \ \models_{\mathbf{Z}} \ r = s \qquad \text{and} \qquad r = s \ \models_{\mathbf{Z}} \ p = q.$$

We say two collections of equations S_1 and S_2 are **Z-equivalent** if

$$\begin{aligned} S_1 \ &\models_{\mathbf{Z}} \ p = q \qquad \text{for } p = q \in S_2 \qquad \text{and} \\ S_2 \ &\models_{\mathbf{Z}} \ p = q \qquad \text{for } p = q \in S_1. \end{aligned}$$

Every equation $p = q$ is $\mathbf{Z}$-equivalent to an equation in the form $s = \mathbf{0}$, namely $p = q$ is $\mathbf{Z}$-equivalent to $p - q = \mathbf{0}$. Thus every equational argument for $\models_{\mathbf{Z}}$ can be put in the form

$$p_1 = \mathbf{0}, \dots, p_k = \mathbf{0} \ \models_{\mathbf{Z}} \ p = \mathbf{0}.$$

Our first lemma says that $\models_{\mathbf{Z}}$ is as strong as $\vdash_{\mathcal{AB}}$.

Lemma 2.1. $S \vdash_{\mathcal{AB}} p = q$ implies $S \models_{\mathbf{Z}} p = q$.

PROOF. For every axiom $r = s$ of $\mathcal{AB}$ one has, for any $\alpha \in \mathcal{I}_{\mathbf{Z}}$, $(\mathbf{Z}, \alpha) \models r = s$ as the axioms are true in the integers under any interpretation of the constants from C. Furthermore for any given $\alpha \in \mathcal{I}_{\mathbf{Z}}$ the property "$(\mathbf{Z}, \alpha) \models$ ____" is preserved by the rules of inference of $\mathcal{AB}$, for example, if $(\mathbf{Z}, \alpha) \models p = q$ then $(\mathbf{Z}, \alpha) \models s[p] = s[q]$. Consequently, if $(\mathbf{Z}, \alpha) \models S$, then for any equation $p = q$ that one can derive from S it follows (by induction on the number of steps in the derivation) that $(\mathbf{Z}, \alpha) \models p = q$, and thus $S \models_{\mathbf{Z}} p = q$. □

Given a sequence of constants $\vec{a} = (a_1, \ldots, a_n)$ from C the **$\vec{a}$-constituents** are the products of the form

$$\widehat{a}_1 \cdots \widehat{a}_n,$$

where each $\widehat{a_i}$ is either a_i or $\mathbf{1} - a_i$.

Lemma 2.2. If $t_1, \ldots, t_{2^n}$ are the 2^n distinct $\vec{a}$-constituents then

$$\begin{array}{llcll} \vdash_{\mathcal{AB}} & t_i^2 & = & t_i & \text{for all } i \\ \vdash_{\mathcal{AB}} & t_i t_j & = & \mathbf{0} & \text{for } i \neq j, \text{ and} \\ \vdash_{\mathcal{AB}} & \mathbf{1} & = & t_1 + \cdots + t_{2^n}. & \end{array}$$

Lemma 2.3. Given $a_1, \ldots, a_n \in C$, for each a_i let $t_1, \ldots, t_{2^{n-1}}$ be the $\vec{a}$-constituents in which a_i is a factor. Then

$$\vdash_{\mathcal{AB}} \ a_i \ = \ t_1 + \cdots + t_{2^{n-1}}.$$

Proof. We have

$$\vdash_{\mathcal{AB}} \ a_i \ = \ [a_1 + (1 - a_1)] \cdots a_i \cdots [a_n + (1 - a_n)]$$

so we just have to suitably multiply out the right side. □

Lemma 2.4. For each term $p(\vec{a})$ there is a sum $\sum m_i t_i$ of $\vec{a}$-constituents t_i, where the m_i are integers, such that

$$\vdash_{\mathcal{AB}} \ p(\vec{a}) \ = \ \sum m_i t_i \ .$$

Proof. Use the replacement rule to replace each a_i in $p(\vec{a})$ by its expansion in $\vec{a}$-constituents as given in Lemma 2.3. Then simplify. □

Lemma 2.5. If $\vdash_{\mathcal{AB}} p(\vec{a}) = \sum m_i t_i$, where the t_i are distinct $\vec{a}$-consitituents and the m_i are integers, then the m_i are Boole's moduli, that is, $\sum m_i t_i$ is the complete expansion of Boole.

Proof. From $\vdash_{\mathcal{AB}} p(\vec{a}) = \sum m_i t_i$ follows, by multiplying both sides by t_i, and by the basic results on constituents in Lemma 2.2, that

$$\vdash_{\mathcal{AB}} \ p(\vec{a}) t_i \ = \ m_i t_i.$$

Then by Lemma 2.1

$$\models_{\mathbf{Z}} \ p(\vec{a}) t_i \ = \ m_i t_i.$$

Choose $\alpha \in \mathcal{I}_{\mathbf{Z}}$ such that $\alpha t_i = 1$. (Of course such an α exists.) Then from the last assertion we have

$$(\mathbf{Z}, \alpha) \ \models \ p(\vec{a}) t_i \ = \ m_i t_i,$$

and thus $\alpha p = m_i$. This means the m_i are the moduli of Boole. □

Let $\sum m_i t_i$ be Boole's complete expansion of $p(\vec{a})$. Then we say that an $\vec{a}$-constituent is **a constituent of** $p(\vec{a})$ if it is one of the t_i with a nonzero coefficient m_i. Let $\mathcal{C}(p(\vec{a}))$ be the collection of constituents of $p(\vec{a})$.

Lemma 2.6. $S \vdash_{\mathcal{AB}} p(\vec{a}) = \mathbf{0}$ iff $S \vdash_{\mathcal{AB}} t = \mathbf{0}$ for $t \in \mathcal{C}(p(\vec{a}))$.

Proof. This follows from Lemmas 2.4 and 2.5. □

Lemma 2.7. $S \models_{\mathbf{Z}} p(\vec{a}) = 0$ iff $S \models_{\mathbf{Z}} t = 0$ for $t \in \mathcal{C}(p(\vec{a}))$.

PROOF. Let $\sum m_i t_i$ be the complete expansion of $p(\vec{a})$. As $\vdash_{\mathcal{AB}} p(\vec{a}) = \sum m_i t_i$ holds by Lemmas 2.4 and 2.5, $\models_{\mathbf{Z}} p(\vec{a}) = \sum m_i t_i$ holds by Lemma 2.1. Then for $\alpha \in \mathcal{I}_{\mathbf{Z}}$ we have

$$(\mathbf{Z}, \alpha) \models p(\vec{a}) = \sum m_i t_i$$

so

$$\mathbf{Z} \models \alpha p = \sum m_i \alpha t_i.$$

Note that at most one αt_i is nonzero. Then for $\alpha \in \mathcal{I}_{\mathbf{Z}}$ with $(\mathbf{Z}, \alpha) \models S$,

$$\mathbf{Z} \models \alpha p = 0 \qquad \text{iff} \qquad \mathbf{Z} \models \alpha t = 0 \qquad \text{for } t \in \mathcal{C}(p),$$

so

$$(\mathbf{Z}, \alpha) \models p = \mathbf{0} \qquad \text{iff} \qquad (\mathbf{Z}, \alpha) \models t = \mathbf{0} \qquad \text{for } t \in \mathcal{C}(p).$$

□

Now we consider an argument $S \ \therefore p = 0$, where the equations in S are assumed to be of the form $r = 0$. Define $\mathcal{C}(S)$, **the constituents of** S, to be the collection of constituents of the various r where $r = 0$ is in S. And let $\mathcal{CE}(S)$ be the collection of **constituent equations** $t = 0$ for t a constituent of S. Let $\vec{a}$ be a list of all the constant symbols from C that appear in S or p.

Lemma 2.8. Using $\vec{a}$-constituents we have:

$$S \vdash_{\mathcal{AB}} p = 0 \qquad \text{iff} \qquad \mathcal{C}(p) \subseteq \mathcal{C}(S).$$

PROOF. Lemma 2.6 implies that S is $\mathcal{AB}$-equivalent to the collection of equations $\mathcal{CE}(S)$. Then

(2) $$S \vdash_{\mathcal{AB}} p = 0 \qquad \text{iff} \qquad \mathcal{CE}(S) \vdash_{\mathcal{AB}} t = 0 \qquad \text{for } t \in \mathcal{C}(p).$$

Now $\mathcal{CE}(S) \vdash_{\mathcal{AB}} t = 0$ holds if $t \in \mathcal{C}(S)$. And if $t \notin \mathcal{C}(S)$ then choose $\alpha \in \mathcal{I}_{\mathbf{Z}}$ such that $\alpha t = 1$. Then $(\mathbf{Z}, \alpha) \models \mathcal{CE}(S)$ but $(\mathbf{Z}, \alpha) \not\models t = 0$, so by Lemma 2.1 we see that $\mathcal{CE}(S) \nvdash_{\mathcal{AB}} t = 0$. Thus

(3) $$\mathcal{CE}(S) \vdash_{\mathcal{AB}} t = 0 \qquad \text{iff} \qquad t \in \mathcal{C}(S).$$

Items (2) and (3) prove the lemma. □

Lemma 2.9. Using $\vec{a}$-constituents we have:

$$S \models_{\mathbf{Z}} p = 0 \qquad \text{iff} \qquad \mathcal{C}(p) \subseteq \mathcal{C}(S).$$

PROOF. By Lemma 2.7, $S \models_{\mathbf{Z}} t = 0$ for each $t \in \mathcal{C}(S)$; and also $\mathcal{CE}(S) \models_{\mathbf{Z}} r = 0$ for each $r = 0$ in S. Thus

(4) $$S \models_{\mathbf{Z}} p = 0 \qquad \text{iff} \qquad \mathcal{CE}(S) \models_{\mathbf{Z}} t = 0 \qquad \text{for } t \in \mathcal{C}(p).$$

Clearly $\mathcal{CE}(S) \models_{\mathbf{Z}} t = 0$ holds if $t \in \mathcal{C}(S)$. And if $t \notin \mathcal{C}(S)$ then one can find an $\alpha \in \mathcal{I}_{\mathbf{Z}}$ with $(\mathbf{Z}, \alpha) \models \mathcal{CE}(S)$ but $(\mathbf{Z}, \alpha) \not\models t = 0$. This means $\mathcal{CE}(S) \not\models_{\mathbf{Z}} t = 0$. Thus

(5) $$\mathcal{CE}(S) \models_{\mathbf{Z}} t = 0 \qquad \text{iff} \qquad t \in \mathcal{C}(S).$$

Combining (4) and (5) proves the lemma. □

The last two lemmas immediately give the following version of Boole's Rule of 0 and 1.

Theorem 2.10. [Boole's Rule of 0 and 1] An equational argument

$$p_1(\vec{a}) = q_1(\vec{a}), \ldots, p_k(\vec{a}) = q_k(\vec{a}) \qquad \therefore p(\vec{a}) = q(\vec{a})$$

has a derivation in the proof system $\mathcal{AB}$ iff every 0–1 interpretation of C that makes the premises true in the ring of integers $\mathbf{Z}$ also makes the conclusion true in $\mathbf{Z}$.

This gives a (truth-table) **decision procedure** for the proof system $\mathcal{AB}$. Also from this theorem we easily have the following (Hailperin's Axiom H10):

Corollary 2.11. $p^2 = \mathbf{0} \vdash_{\mathcal{AB}} p = \mathbf{0}$.

3. Boole's Main Theorems

Expansion Theorem

Any term $p(\vec{a}, \vec{b}\,)$ has an expansion on the symbols $\vec{a}$ given by

$$p(\vec{a}, \vec{b}\,) \;=\; p(\mathbf{1}, \ldots, \mathbf{1}, \vec{b}\,)a_1 \cdots a_n \;+\; \cdots \;+\; p(\mathbf{0}, \ldots, \mathbf{0}, \vec{b}\,)(\mathbf{1} - a_1) \cdots (\mathbf{1} - a_n).$$

This follows immediately from the Rule of 0 and 1.

It is worth noting that we do not require that all the constant symbols mentioned in the term $p(\vec{a}, \vec{b})$ actually occur in p. Thus, for example, we could expand $a + b$ on the constant symbols a, c if we wished.

This theorem expresses the term $p(\vec{a}, \vec{b}\,)$ as a sum $\sum q_i t_i$, where the t_i are $\vec{a}$-constituents, and the symbols from $\vec{a}$ do not appear in the terms q_i. From such an expansion $p = \sum q_i t_i$ it is easy to see that p is idempotent iff each of the coefficients q_i is idempotent.

Reduction Theorem

A system of equations

$$\begin{cases} p_1 & = & \mathbf{0} \\ & \vdots & \\ p_k & = & \mathbf{0} \end{cases}$$

is equivalent to the single equation

$$p_1^2 + \cdots + p_k^2 \;=\; \mathbf{0}.$$

Let $\vec{a}$ be the list of all the constant symbols of C appearing in the various p_i. Then the proof of the theorem follows from complete expansions of the p_i on the symbols $\vec{a}$, and observing that squaring a complete expansion simply squares all the coefficients of the expansion. Thus each p_i^2 will have only nonnegative coefficients in its complete expansion on $\vec{a}$, and this guarantees that the $\vec{a}$-constituents of $p_1^2 + \cdots + p_k^2$ will be precisely the $\vec{a}$-constitutents that belong to at least one of the p_i.

Elimination Theorem

The result of eliminating the constant symbols $a_1, \ldots, a_m$ from

$$p(\vec{a}, \vec{b}\,) \;=\; \mathbf{0}$$

is

$$q(\vec{b}\,) \;=\; \mathbf{0},$$

where

$$q(\vec{b}\,) \;=\; p(\mathbf{1}, \ldots, \mathbf{1}, \vec{b}\,) \;\cdots\; p(\mathbf{0}, \ldots, \mathbf{0}, \vec{b}\,).$$

Proof. The fact that $q(\vec{b}\,) = \mathbf{0}$ follows from $p(\vec{a}, \vec{b}\,) = \mathbf{0}$ in $\mathcal{AB}$ is an easy application of the Rule of 0 and 1. But we need to explain what Boole means when he says (1854, p. 8) that the equation $q(\vec{b}\,) = \mathbf{0}$ expresses

> . . . the whole amount of relation implied by the premises among the elements which we wish to retain.

Unfortunately Boole does not clarify this, nor justify that his elimination procedure accomplishes this. The simplest explanation of what he means is that if $r(\vec{b}\,) = \mathbf{0}$ is any equation that follows from $p(\vec{a}, \vec{b}\,) = \mathbf{0}$ then $r(\vec{b}\,) = \mathbf{0}$ follows from $q(\vec{b}\,) = \mathbf{0}$, i.e., if $p(\vec{a}, \vec{b}\,) = \mathbf{0} \vdash_{\mathcal{AB}} r(\vec{b}\,) = \mathbf{0}$ then $q(\vec{b}\,) = \mathbf{0} \vdash_{\mathcal{AB}} r(\vec{b}\,) = \mathbf{0}$.

This is indeed true, and thus $q(\vec{b}\,) = \mathbf{0}$ is the strongest possible equation that one can deduce about the symbols $\vec{b}$ from the premiss $p(\vec{a}, \vec{b}\,) = \mathbf{0}$.

To see that this is true, first observe that since $r(\vec{b}\,) = \mathbf{0}$ is $\mathcal{AB}$-equivalent to the system of $t(\vec{b}\,) = \mathbf{0}$, where t is a $\vec{b}$-constituent of $r(\vec{b}\,)$, it suffices to show the above when $r(\vec{b}\,)$ is a $\vec{b}$-constituent $t(\vec{b}\,)$.

So we suppose $p(\vec{a}, \vec{b}\,) = \mathbf{0} \vdash_{\mathcal{AB}} t(\vec{b}\,) = \mathbf{0}$, where $t(\vec{b}\,)$ is a $\vec{b}$-constituent. Carry out a partial expansion of p on the constant symbols $\vec{b}$ to obtain

$$\vdash_{\mathcal{AB}} \; p(\vec{a}, \vec{b}\,) \; = \; \sum_{i=1}^{k} r_i(\vec{a}) t_i(\vec{b}\,), \tag{6}$$

where the t_i are $\vec{b}$-constituents. Choose a 0–1 interpretation α of C such that $\alpha t = 1$. (This is of course possible.) Under this interpretation $t = \mathbf{0}$ is false, so by the Rule of 0 and 1, since $p = \mathbf{0} \vdash_{\mathcal{AB}} t = \mathbf{0}$, we must have $\alpha p \neq 0$. Now

$$\alpha p \; = \; \sum_{i=1}^{k} (\alpha r_i)(\alpha t_i),$$

so there must be some i such that $t_i = t$, say $t_j = t$ (for otherwise $\alpha p = 0$). Then

$$\alpha p \; = \; \alpha r_j.$$

Now r_j only involves the symbols $\vec{a}$ from C, and since α could map those arbitrarily to 0,1 and have the desired property, it follows that, under every 0-1 interpretation β of C, $\beta r_j \neq 0$. Thus from (6) the constituent t belongs to every term $p(e_1, \ldots, e_m, \vec{b}\,)$, where each e_i is one of $\mathbf{0}$ or $\mathbf{1}$, and consequently it belongs to the product of all such terms, which is $q(\vec{b}\,)$. Thus from $q = \mathbf{0}$ we can derive $t = \mathbf{0}$ in $\mathcal{AB}$, which was to be shown. □

This is not the end of the story on the Elimination Theorem. Jevons did not discuss it in his modification of Boole's methods, but in the 1890's Schröder highlighted it in his famous volumes, Algebra der Logik. Schröder proved that the Elimination Theorem held for Boolean algebra. His version was that $ax + bx' = 0$ holds iff $ab = 0$ and, for some u, $x = bu' + a'u$.

Then Schröder ventured beyond equations, into negated equations, and considered systems of the form:

$$\begin{array}{rclc} p_i(\vec{a}, \vec{b}\,) & = & \mathbf{0} & \qquad 1 \le i \le k \\ p_i(\vec{a}, \vec{b}\,) & \neq & \mathbf{0} & \qquad k+1 \le i \le m \end{array}$$

and found that he could do very little to formulate a satisfying elimination theorem, so he posed it as a challenge problem.

In 1919 Skolem was able to cast this as problem of **quantifier-elimination** for the first-order theory of power set Boolean algebras, and gave a beautiful treatment. One can view Boole's Elimination Theorem, as formulated in the first-order theory of Boolean algebra, as the following theorem:

$$\Big[\exists x_1 \cdots \exists x_m \Big(p(\vec{x}, \vec{y}\,) = \mathbf{0}\Big)\Big] \longleftrightarrow \Big[q(\vec{y}\,) = \mathbf{0}\Big],$$

where the q is as defined in Boole's Elimination Theorem. Thus we can think of Boole's work on elimination as the beginning of one of the most popular methods of the 20th century to prove decidability results, namely the **elimination of quantifiers.**

4. Boole's Method

We summarize Boole's method of using equations to handle arguments in logic:

Step 1. Given some propositions in logic as premises, **translate** them into equational form.

Step 2. Combine the equations into a single equation using the **Reduction Theorem**.

Step 3. Apply the **Elimination Theorem** to obtain the most general conclusion in the desired variables.

Step 4. Apply the **Expansion Theorem** to give the conclusion as a collection of equations of the form $t = 0$, t being a constituent.

Step 5. Interpret the equations $t = 0$ as propositions.

5. A More Sophisticated Approach

We can make the proof of the Rule of 0 and 1 much shorter than that given in the last section by appealing to results from the theory of rings.

Let $\mathcal{R}_{\mathcal{AB}}$ denote the class $(\mathbf{R}, \alpha)$ of additively nonnilpotent commutative rings $\mathbf{R}$ with a mapping $\alpha : C \to R$ such that each αc is idempotent, and $\alpha(C)$ generates $\mathbf{R}$. From Theorem 10.1 of Pierce [**17**] on sheaf representations[18] it follows that for $(R, \alpha) \in \mathcal{R}_{\mathcal{AB}}$ we have $\mathbf{R}$ isomorphic to a subdirect power of the ring of integers $\mathbf{Z}$ as the stalks of a reduced sheaf for $\mathbf{R}$ are indecomposable additively nonnilpotent commutative rings generated by idempotents.

We say

$$(\mathbf{R}, \alpha) \models p = q$$

if the interpretation of the constant symbols given by α makes $p = q$ true in $\mathbf{R}$. For $S \cup \{p = q\}$ a collection of equations define

$$S \models_{\mathcal{AB}} p = q$$

to mean that, for every member $(\mathbf{R}, \alpha)$ of $\mathcal{R}_{\mathcal{AB}}$, we have

$$(\mathbf{R}, \alpha) \models S \text{ implies } (\mathbf{R}, \alpha) \models p = q$$

The following is the basic soundness and completeness theorem for $\mathcal{AB}$. The proof is just like the usual proof for the completeness of equational logic based on Birkhoff's rules except that we do not have to be concerned with the substitution rule.

Theorem 5.1. $S \vdash_{\mathcal{AB}} p = q$ iff $S \models_{\mathcal{AB}} p = q$.

Proof. ($\Longrightarrow$) We suppose $S \vdash_{\mathcal{AB}} p = q$, and let $(\mathbf{R}, \alpha) \in \mathcal{R}_{\mathcal{AB}}$ be such that $(\mathbf{R}, \alpha) \models S$. If $p = q$ is an axiom of $\mathcal{AB}$, or is in S, or is an equation arising from a single application of any of the rules of inference of $\mathcal{AB}$ to S, then it is easy to check that $(\mathbf{R}, \alpha) \models p = q$. Then, by a simple induction proof on the length of a derivation, $(\mathbf{R}, \alpha) \models p = q$. Consequently $S \models_{\mathcal{AB}} p = q$, and this proves $\mathcal{AB}$ is **sound**.

($\Longleftarrow$) Now we suppose $S \nvdash_{\mathcal{AB}} p = q$. Let T be the set of terms in the language of $\mathcal{AB}$, and let $\mathbf{T}$ be the corresponding **term algebra**. Define an equivalence relation $\equiv$ on T by

$$s \equiv t \quad \text{iff} \quad S \vdash_{\mathcal{AB}} s = t$$

[18] From this sheaf representation it is not difficult to show that $\mathbf{R}$ must be isomorphic to a Boolean power $\mathbf{Z}[\mathbf{B}]^*$ of $\mathbf{Z}$.

Then $\equiv$ gives a congruence on $\mathbf{T}$, and the quotient $(\mathbf{T}/{\equiv}, \nu)$ is in $\mathcal{R}_{\mathcal{AB}}$, where the interpretation ν is the natural one given by $\nu(c) = c/{\equiv}$.[19] $(\mathbf{T}/{\equiv}, \nu)$ satisfies S but not $p = q$, so we can conclude that $S \not\models_{\mathcal{AB}} p = q$. This proves that $\mathcal{AB}$ is **complete**. □

Now we are ready for a second proof of the Rule of 0 and 1.

Theorem 5.2. $S \vdash_{\mathcal{AB}} p = q$ iff $S \models_{\mathbf{Z}} p = q$.

Proof. ($\Longrightarrow$) If $S \vdash_{\mathcal{AB}} p = q$ then by the soundness of $\mathcal{AB}$ we have $S \models_{\mathcal{AB}} p = q$, and thus in particular $S \models_{\mathbf{Z}} p = q$.

($\Longleftarrow$) Now we suppose $S \nvdash_{\mathcal{AB}} p = q$. Then, from the proof of the completeness theorem, $(\mathbf{T}/{\equiv}, \nu)$ satisfies S but not $p = q$. By earlier remarks we have $(\mathbf{T}/{\equiv}, \nu)$ is a subdirect product of $(\mathbf{Z}, \alpha)$'s. By taking a projection on a suitable coordinate we find that for some α we have $(\mathbf{Z}, \alpha) \models S$ but $(\mathbf{Z}, \alpha) \not\models p = q$. Thus $S \not\models_{\mathbf{Z}} p = q$. This finishes the proof. □

6. A Remark on Hailperin's System

Regarding the proof of the Expansion Theorem, Hailperin says that he has not assumed Boole's Rule of 0 and 1,[20] and so he must give a different proof ([**6**] p. 144).

> We first establish that Boole's Law of Development holds for any Boolean multiset term. Our proof is different from Boole's since we can't use his assumption that an equation is established if it holds for all assignments of 0 and 1 to the variables—in fact, contrariwise, we shall prove this is so by using the Law of Development ...

His version of the Expansion Theorem is as follows ([**6**] page 95).

> Theorem 2.33 (Law of Development). *If $f(x)$ is any term and x a Boolean variable, then*
>
> $$f(x) \;=\; f(1)x \;+\; f(0)(1-x),$$
>
> *where $f(1)$ is the term obtained from $f(x)$ by replacing x throughout by 1 and, similarly, $f(0)$ is obtained by replacing x by 0.*
>
> Proof. Clearly $f(1)$ and $f(0)$ are terms. From the idempotency of x and simple ring properties one sees that $f(x)$ (which may have other variables present) is equal to a linear form, i.e., $f(x) = Ax + B$ where A and B do not involve x, and thus $f(1) = A + B$ and $f(0) = B$. The result is now immediate since $Ax + B = (A + B)x + B(1 - x)$.

At this point he is clearly applying substitution (called replacement) to the variable x. Thus there appears to be a substitution rule in his logic that allows, at the minimum, the substitution of 0's and 1's for the lower case variables for idempotents. When working with ground terms one can directly prove (by induction on the length of derivation) that if $S(\vec{a}) \vdash_{\mathcal{AB}} p(\vec{a}) = q(\vec{a})$ then one can indeed substitute for the a_i any idempotent terms e_i, i.e., $S(\vec{e}) \vdash_{\mathcal{AB}} p(\vec{e}) = q(\vec{e})$ holds. For the case that one is proving the Expansion Theorem when $+$ is treated as union one can find just such a proof in Schröder's Vol. I (1890), §19. However no such proof is indicated in Hailperin's book.

Since, as we said before, Boole's literal symbols refer to fixed classes throughout a given discussion, there does not seem to be any principle in Boole's work that allows such a substitution other than his Rule of 0 and 1, and this rule only permits the substitution of 0's and 1's for literal symbols. Consequently we believe the system $\mathcal{AB}$ is more faithful to the intention of Boole's work; and it has the substitution rule, of idempotent terms for symbols from C, as a consequence of the Rule of 0 and 1.

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[19] If $S = \emptyset$ then the ring $\mathbf{T}/{\equiv}$ is the quotient of the commutative ring of polynomials $\mathbf{Z}[C]$ by the ideal generated by the polynomials $c^2 - c$, i.e., this is the commutative ring presented by $\langle C; \{c^2 - c = \mathbf{0} : c \in C\}\rangle$.

[20] Hailperin has the Rule of 0 and 1 referring only to equations, and not to arguments. That is, he has the Rule for $\therefore p = q$, when the Rule actually applies more generally to $p_1 = q_1, \ldots, p_k = q_k \;\therefore p = q$.

7. Tying the Proof System to the Real World

Even though the Rule of 0 and 1 is applicable to the proof system $\mathcal{AB}$ it still remains to show that an argument

$$p_1 = q_1, \ \ldots\ , \ p_k = q_k \qquad \therefore p = q \tag{7}$$

has a derivation in $\mathcal{AB}$ precisely when the interpretation as an argument about classes is correct. In order to be able to interpret (7) as an argument about classes we need the terms that appear in the argument to be interpretable as classes when the constant symbols are interpreted as classes. Boole preferred to have the premiss and conclusion equations in such interpretable form. One could then see that the equational argument indeed captured the precise meaning of an argument involving categorical propositions, or more general propositions.

But then Boole allowed noninterpretable terms to be used in the intermediate steps of a derivation of the conclusion, and this created a very uneasy feeling among his successors. We want to show that indeed his results give correct arguments in the real world of working with classes. *And it is precisely this connection that seems beyond the methods of the 19th century.* The following diagram indicates the facets of an equational argument (7) that we are considering:

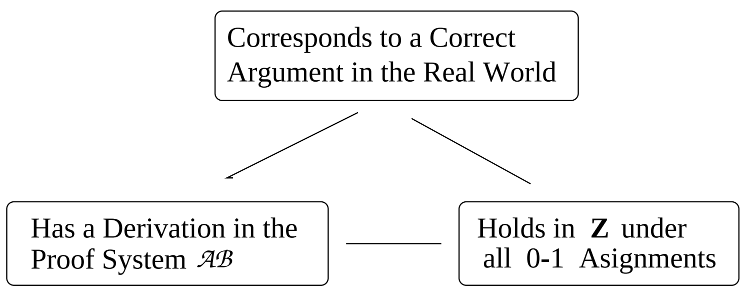

Fig. **53** Three Ways of Viewing an Equational Argument

We have proved that the bottom two are equivalent, and now we want to show that these are equivalent to the top.

We will start by defining the notion of an interpretable term. Of course it means that one is working with terms where one never adds subterms that correspond to overlapping classes, and one never subtracts one term from another unless its interpretation is contained in the latters interpretation.

Directly Interpretable

Given a domain, or universe, U we want to describe those terms that are always *directly* interpretable (without the help of algebraic manipulations) when the constants from C are assigned to subclasses of U. The condition that is needed is that all subterms have interpretations. For example the term $(a+b)-ab$ is not directly interpretable in U as $a+b$ is not always interpretable. (However it is equivalent to the term $a+(b-ab)$ which is directly interpretable.)

The collection of terms directly interpretable in U will be called $\mathcal{D}$. An interpretation in U starts with an interpretation λ of the constants from C as subclasses of U, i.e., $\lambda c \subseteq U$ for $c \in C$. Then we want to extend the domain of λ to include the directly interpretable terms. The collection of all such λ will be denoted Λ.

Although the notion of directly interpretable terms and interpretations is intuitively clear, nonetheless writing out a precise definition is somewhat involved. This proceeds by simultaneous definition of the members of $\mathcal{D}$ and the extension of members of Λ. One difficulty that we have to deal with is that if a term p is directly interpretable then most likely $-p$ is not. To handle this we will work with the binary operation of **subtraction**, e.g., $p - q$, rather than the unary operation of *minus*. Here is the definition of the collection $\mathcal{D}$ of **directly interpretable terms**:

- c is in $\mathcal{D}$ for $c \in C$.
- $\mathbf{0}$ and $\mathbf{1}$ are in $\mathcal{D}$, and $\lambda\mathbf{0} = \emptyset$ and $\lambda\mathbf{1} = U$ for $\lambda \in \Lambda$.

  For $s, t \in \mathcal{D}$:
- $s \cdot t \in \mathcal{D}$, and $\lambda(s \cdot t) = \lambda s \cap \lambda t$ for $\lambda \in \Lambda$.
- If $\lambda s \cap \lambda t = \emptyset$ for all $\lambda \in \Lambda$ then $s + t \in \mathcal{D}$, and $\lambda(s + t) = \lambda s \cup \lambda t$ for $\lambda \in \Lambda$.
- If $\lambda s \supseteq \lambda t$ for all $\lambda \in \Lambda$ then $s - t \in \mathcal{D}$, and $\lambda(s - t) = \lambda s \setminus \lambda t$ for $\lambda \in \Lambda$.

Note that terms p of the form $\sum t_i$, with the t_i distinct constituents, are in $\mathcal{D}$, and $\lambda p = \sum \lambda t_i$.

An equation $p = q$, or an argument as in (7), with all terms mentioned being in $\mathcal{D}$, is said to be **directly interpretable**. If an argument (7) is directly interpretable then it is clear that, for $\lambda \in \Lambda$, the interpretation using λ gives an argument about classes, namely

$$\lambda p_1 = \lambda q_1, \ \ldots, \ \lambda p_k = \lambda q_k \qquad \therefore \ \lambda p = \lambda q. \tag{8}$$

This says that if the class λp_i equals the class λq_i for all i then the class λp equals the class λq. We will say that a directly interpretable argument (7) is **valid in** U if (8) is a correct argument for all interpretations $\lambda \in \Lambda$. We write

$$p_1 = q_1, \ \ldots, \ p_k = q_k \ \models_U \ p = q$$

to assert that (7) is valid in U.

Interpreting in $\mathbf{Z}^U$

We make the connection between Boole's methods $\vdash_{\mathcal{AB}}$ and $\models_{\mathbf{Z}}$ and the real world $\models_U$ through Hailperin's model $\mathbf{Z}^U$ of signed multisets. *This step seems unlikely to have been available to 19th century logicians.* Using $\mathbf{Z}^U$ is somewhat analogous to the creation of ideal numbers, which was done in the last century, but there is no indication that logicians in the 19th century even considered the use of ideal classes to justify Boole's work.

Let $\mathcal{I}_{\mathbf{Z}^U}$ be the collection of interpretations of the elements of C as *idempotent* elements of $\mathbf{Z}^U$. Then each $\varphi \in \mathcal{I}_{\mathbf{Z}^U}$ can be extended inductively to interpret all terms as elements of $\mathbf{Z}^U$:

- $\varphi\mathbf{0} = \mathbf{0} \qquad \varphi\mathbf{1} = \mathbf{0}$
- $\varphi(-s) = -\varphi s$
- $\varphi(s + t) = \varphi s + \varphi t$
- $\varphi(s \cdot t) = \varphi s \cdot \varphi t$

For $\varphi \in \mathcal{I}_{\mathbf{Z}^U}$ we say $(\mathbf{Z}^U, \varphi)$ **satisfies** $p = q$, written $(\mathbf{Z}^U, \varphi) \models p = q$, if $p = q$ is true in $\mathbf{Z}^U$ under the interpretation φ. This means $\varphi p = \varphi q$.

An argument (7) is said to be **valid** in $\mathbf{Z}^U$, written

$$p_1 = q_1, \ \ldots, \ p_k = q_k \ \models_{\mathbf{Z}^U} \ p = q,$$

if for every $\varphi \in \mathcal{I}_{\mathbf{Z}^U}$ such that the equations $\varphi p_i = \varphi q_i$ are true in $\mathbf{Z}^U$ the equation $\varphi p = \varphi q$ is also true in $\mathbf{Z}^U$.

Making the Connections

In the following let $p_1 = q_1, \ \ldots, \ p_k = q_k \ \therefore p = q$ be an argument that is directly interpretable in U.

Lemma 7.1. The following are equivalent:

$$p_1 = q_1, \ \ldots, \ p_k = q_k \quad \models_U \quad p = q$$
$$p_1 = q_1, \ \ldots, \ p_k = q_k \quad \models_{\mathbf{Z}^U} \quad p = q.$$

Proof. For $\lambda \in \Lambda$ define $\varphi_\lambda \in \mathcal{I}_{\mathbf{Z}^U}$ by

$$\varphi_\lambda c = \mathcal{X}_{\lambda c}.$$

Then the mapping $\lambda \mapsto \varphi_\lambda$ is a bijection from Λ to $\mathcal{I}_{\mathbf{Z}^U}$. Furthermore, one has for each directly interpretable term s

$$\varphi_\lambda s = \mathcal{X}_{\lambda s}.$$

This is proved by induction on $s \in \mathcal{D}$. Thus we have, for any two terms r and s,

$$\lambda r = \lambda s \qquad \text{iff} \qquad \varphi_\lambda r = \varphi_\lambda s$$

as

$$\lambda r = \lambda s \qquad \text{iff} \qquad \mathcal{X}_{\lambda r} = \mathcal{X}_{\lambda s} \qquad \text{iff} \qquad \varphi_\lambda r = \varphi_\lambda s.$$

So for any equation $r = s$,

$$(\mathbf{Z}, \lambda) \models r = s \qquad \text{iff} \qquad (\mathbf{Z}^U, \varphi_\lambda) \models r = s.$$

This suffices to prove the lemma as every $\varphi \in \mathcal{I}_{\mathbf{Z}^U}$ is a φ_λ. □

Lemma 7.2. The following are equivalent:

(9) $$p_1 = q_1,\ \ldots,\ p_k = q_k \quad \models_{\mathbf{Z}} \quad p = q$$

(10) $$p_1 = q_1,\ \ldots,\ p_k = q_k \quad \models_{\mathbf{Z}^U} \quad p = q.$$

Proof. For $u \in U$ and $\varphi \in \mathcal{I}_{\mathbf{Z}^U}$ define $\varphi_u \in \Lambda$ by $\varphi_u(c) = (\varphi c)(u)$.

Suppose (9) holds. For $\varphi \in \mathcal{I}_{\mathbf{Z}^U}$, if the equations $\varphi p_i = \varphi q_i$ are true in $\mathbf{Z}^U$ then the equations $\varphi_u p_i = \varphi_u q_i$ are true in $\mathbf{Z}$ for each $u \in U$, so by (9) $\varphi_u p = \varphi_u q$ is true in $\mathbf{Z}$. But then $\varphi p = \varphi q$ in $\mathbf{Z}^U$. This means (10) holds.

And if (9) does not hold then choose a $\lambda \in \Lambda$ such that the equations $\lambda p_i = \lambda q_i$ hold in $\mathbf{Z}$ but $\lambda p = \lambda q$ does not. Define $\varphi \in \mathcal{I}_{\mathbf{Z}^U}$ by

$$\varphi c = \begin{cases} \mathbf{0} & \text{if } \lambda c = 0 \\ \mathbf{1} & \text{if } \lambda c = 1 \end{cases}$$

for $c \in C$. Then the equations $\varphi p_i = \varphi q_i$ hold in $\mathbf{Z}^U$ as $(\varphi p_i)(u) = \lambda p_i$ and $(\varphi q_i)(u) = \lambda q_i$ for all $u \in U$, but $\varphi p = \varphi q$ does not hold as $(\varphi p)(u) \neq (\varphi q)(u)$ for any $u \in U$. Thus (10) does not hold in $\mathbf{Z}^U$. □

Combining the previous two lemmas gives the main result:

Theorem If $S \ \therefore\ p = q$ is directly interpretable in U then it is valid in U iff $S \vdash_{\mathcal{AB}} p = q$ iff $S \models_{\mathbf{Z}} p = q$.

8. The Connection with Boolean Rings

$\mathcal{BR}$ is the equational proof system for Boolean rings given in Appendix 1. We want to modify this so that it can be applied to the equational arguments analyzed by Boole. This could be done by using $\mathcal{AB}$ augmented by the axioms $s + s = \mathbf{0}$ for s any term. However we will choose another route that stays closer to the setup of $\mathcal{BR}$.

Let $\mathcal{BR}[C]$ be the expansion of $\mathcal{BR}$ by the set C of constants as in Appendix 1. In the following we say that a ground term $p(a_1, \ldots, a_n)$ is **$\mathcal{B}$-idempotent** if it is idempotent in Boole's system, that is, all the moduli $p(\mathbf{1}, \ldots, \mathbf{1}), \ldots, p(\mathbf{0}, \ldots, \mathbf{0})$ are either 0 or 1 when evaluated in $\mathbf{Z}$. Of course directly interpretable terms are $\mathcal{B}$-idempotent.

A ground equational argument

(11) $$p_1 = q_1,\ \ldots,\ p_k = q_k \qquad \therefore p = q$$

has **$\mathcal{B}$-idempotent terms** if all the p's and q's are $\mathcal{B}$-idempotent. The Rule of 0 and 1 says that we can use the integers $\mathbf{Z} = (Z, +, \cdot, -, 0, 1)$ to test if the equational argument (11) has a derivation in $\mathcal{AB}$, namely by checking if the argument holds in $\mathbf{Z}$ under all possible 0–1 interpretations of the constants in C.

Let $\mathbf{Z}_2$ be the two-element Boolean ring, the ring of integers modulo 2. Then for α any 0–1 interpretation of C and for any $\mathcal{B}$-idempotent ground term h the value of $\alpha(h)$ in $\mathbf{Z}$ is the same as in $\mathbf{Z}_2$. But this means that for an argument with $\mathcal{B}$-idempotent terms one can use 0–1 interpretations in $\mathbf{Z}_2$ and carry out the calculations in $\mathbf{Z}_2$, rather than in $\mathbf{Z}$, to apply the Rule of 0 and 1 to determine if the argument is valid.

As we also have a Rule of 0 and 1 for $\mathcal{BR}[C]$, it follows that for arguments (11) with $\mathcal{B}$-idempotent terms we have

$$p_1 = q_1,\ \ldots,\ p_k = q_k \ \vdash_{\mathcal{AB}}\ p = q$$
$$\text{iff}$$
$$p_1 = q_1,\ \ldots,\ p_k = q_k \ \vdash_{\mathcal{BR}[C]}\ p = q$$

Thus a natural modernization of Boole's system is the ground equational theory of Boolean rings augmented by a set C of constants. But we want to emphasize that Boole never used the two-element Boolean ring $\mathbf{Z}_2$ to test the validity of arguments, only the ordinary number system.

9. The Connection with Boolean Algebra

In order to make the connection with Boolean algebra let us show the connection between Boolean algebra and Boolean rings. (See [**3**], Chapter IV.)

We can translate terms from one system to the other. Define a ground term $\overline{p}$ in the language of $\mathcal{BR}[C]$ for each ground term p in the language of $\mathcal{BA}[C]$ as follows:

- $\overline{\mathbf{0}} = \mathbf{0}$ and $\overline{\mathbf{1}} = \mathbf{1}$
- $\overline{c} = c$ for $c \in C$
- $\overline{s'} = 1 - \overline{s}$
- $\overline{s \vee t} = \overline{s} + \overline{t} + \overline{s} \cdot \overline{t}$
- $\overline{s \wedge t} = \overline{s} \cdot \overline{t}$

And we define a translation from $\mathcal{BR}[C]$ terms to $\mathcal{BA}[C]$ terms:

- $\overline{\mathbf{0}} = \mathbf{0}$ and $\overline{\mathbf{1}} = \mathbf{1}$
- $\overline{c} = c$ for $c \in C$
- $\overline{-s} = \overline{s}$
- $\overline{s + t} = (\overline{s} \wedge \overline{t}') \vee (\overline{s}' \wedge \overline{t})$
- $\overline{s \cdot t} = \overline{s} \wedge \overline{t}$

With this translation one has $\vdash_{\mathcal{BA}[C]} p = \overline{\overline{p}}$ for p a $\mathcal{BA}[C]$ term, and also $\vdash_{\mathcal{BR}[C]} p = \overline{\overline{p}}$ for p a $\mathcal{BR}[C]$ term. Furthermore, arguments that have derivations translate to arguments that have derivations:

$$p_1 = q_1,\ \ldots,\ p_n = q_n \ \vdash_{\mathcal{BR}[C]}\ p = q$$
$$\text{iff}$$
$$\overline{p}_1 = \overline{q}_1,\ \ldots,\ \overline{p}_n = \overline{q}_n \ \vdash_{\mathcal{BA}[C]}\ \overline{p} = \overline{q}$$

as well as:

$$p_1 = q_1,\ \ldots,\ p_n = q_n \ \vdash_{\mathcal{BA}[C]}\ p = q$$
$$\text{iff}$$
$$\overline{p}_1 = \overline{q}_1,\ \ldots,\ \overline{p}_n = \overline{q}_n \ \vdash_{\mathcal{BR}[C]}\ \overline{p} = \overline{q}\ .$$

Thus another modernization of Boole's system is Boolean algebra expanded by a set C of constants. However this is not so natural as $\mathcal{BR}[C]$ because the laws of Boolean algebra do not include the laws of ordinary algebra, for example, $x + (-x) = \mathbf{0}$. And one has a somewhat cumbersome translation between an argument with $\mathcal{B}$-idempotent terms in $\mathcal{AB}$ into an argument in $\mathcal{BA}[C]$. $\mathcal{BA}[C]$, in the case of ground equations, is essentially what Jevons developed, but without the Rule of 0 and 1, and hence without using the two-element Boolean algebra to test for validity.

10. Boole's Use of Division

The main situation where Boole uses division is in finding the general solution for a given a ground equation

$$p(a, b_1, \ldots, b_n) = \mathbf{0}$$

By the Expansion Theorem one easily converts this into an equivalent equation of the form

$$g(b_1, \ldots, b_n)a = h(b_1, \ldots, b_n)$$

Let us, instead of 'dividing', expand each of g and h in terms of the $\vec{b}$-constituents t_i, say as:

$$\begin{aligned} g(b_1, \ldots, b_n) &= \sum m_i t_i \\ h(b_1, \ldots, b_n) &= \sum n_i t_i \ . \end{aligned}$$

Then

$$\sum m_i a t_i = \sum n_i t_i \ ,$$

and thus, for all i,

$$m_i a t_i = n_i t_i \ .$$

Let us look at various cases:

$$\begin{array}{llll} \text{if} & m_i \neq n_i \neq 0 & \text{then} & t_i = \mathbf{0} \\ \text{if} & m_i \neq n_i = 0 & \text{then} & a t_i = \mathbf{0} \\ \text{if} & m_i = n_i \neq 0 & \text{then} & a t_i = t_i \\ \text{if} & m_i = n_i = 0 & \text{then} & \text{no restriction on } a t_i \ . \end{array}$$

Combining these facts we see that any solution for a must be of the following form, where each v_i is a new variable:

$$a = \sum_{m_i = n_i \neq 0} t_i \ + \sum_{m_i = n_i = 0} v_i t_i \ ,$$

and we have the side constraints:

$$t_i = \mathbf{0} \qquad \text{whenever} \qquad m_i \neq n_i \neq 0 \ .$$

It is easy to check that the above expression for a always yields a solution.

Letting $\vec{k}_i$ be the (unique) sequence of 0's and 1's for which $t_i(\vec{k}_i) = 0$ we have

$$\begin{aligned} m_i &= g(\vec{k}_i) \\ n_i &= h(\vec{k}_i) \end{aligned}$$

and thus the result

$$a = \sum_{g(\vec{k}_i) = h(\vec{k}_i) \neq 0} t_i \ + \sum_{g(\vec{k}_i) = h(\vec{k}_i) = 0} v_i t_i \ ,$$

with the side constraints

$$t_i = \mathbf{0} \qquad \text{whenever} \qquad g(\vec{k}_i) \neq h(\vec{k}_i) \neq 0 \ .$$

This is easily checked to be equivalent to Boole's conditions on

$$\frac{h(\vec{k}_i)}{g(\vec{k}_i)}$$

the latter being the result of dividing the equation $ga = h$ to obtain

$$a = \frac{h(b_1, \ldots, b_n)}{g(b_1, \ldots, b_n)}$$

and then formally applying the Expansion Theorem to the right hand side, yielding

$$\frac{h(b_1, \ldots, b_n)}{g(b_1, \ldots, b_n)} = \sum \frac{h(\vec{k}_i)}{g(\vec{k}_i)}\, t_i$$

Appendix 4: Existential Import

In 1880 Peirce [**16**] (page 22) takes strong exception to Boole's use of equations to express particular statements:

> The two kinds of propositions ['All A is B' and 'Some A is B'] are essentially different, and every attempt to reduce the latter to the former must fail. Boole attempts to express 'some men are not mortal', in the form 'whatever men have an unknown character v are not mortal.' But the propositions are not identical, for the latter does not imply that some men have the character v, and accordingly, from Boole's proposition we may legitimately infer that 'whatever mortals have the unknown character v are not men; yet we cannot reason from 'some men are not mortal' to 'some mortals are not men'.

Peirce is evidently starting with Boole's equational representation $vx = v(1-y)$ of 'some men are not mortal' and deriving $vy = v(1-x)$. But the interpretation that 'some mortals are not men' appears to be a careless mistake on his part. Boole does not claim that the equation $vx = v(1-y)$ by itself has existential import, but rather it comes from the fact that he has an explicit side condition, namely that vx is not empty. Indeed Peirce's claim that any equational approach to expressing particular propositions must fail is simply wrong.

Schröder follows Peirce in the attack, but is more careful, saying that the equation $vx = vy$ and side condition 'some v is x' can be simply replaced by the condition $xy \neq 0$. And the latter condition he says cannot be expressed by any equation $p(x, y, z, \dots) = 0$. Schröder is working with Boolean algebra and the modern semantics, and not Boole's algebra of logic and restricted semantics, but his argument carries over. All he does is expand p in terms of x, y-constituents to obtain

$$p(x, y, z, \dots) \;=\; p_1 xy \;+\; p_2 \overline{x} y \;+\; p_3 x \overline{y} \;+\; p_4 \overline{x}\overline{y}$$

and by interpreting x and y as the empty class or universe in various ways it follows that all the p_i must be 0 in order to capture the meaning of $xy \neq 0$. (One can modify this argument to work with the restricted semantics, namely by letting x and y both be p_1 one can show p_1 must be 0, etc.)

So Schröder is correct in saying that one cannot capture existential import on an 'equation by equation' basis. But he overlooks the possibility that for the purpose of arguments one can use equations to syntactically capture the existential import. This only requires a small amount of care in setting up the equations, using a slightly different system with the modern semantics than with the restricted semantics.

We will develop the method by starting with Schröder's observation that $ab \neq 0$ does indeed precisely capture the meaning of 'Some a is b', and following Schröder, work in the general setting of equations and negated equations. Thus we can take for our premises a set of ground equations and negated equations:

$$\mathcal{S} \;=\; \{p_1 = 0,\; \dots,\; p_k = 0\} \;\cup\; \{q_1 \neq 0,\; \dots,\; q_m \neq 0\}$$

We want to show how we can derive any consequence of the form $p = 0$ or of the form $q \neq 0$ using only equational logic.

For convenience let C_W be an infinite collection of constants contained in C. In the case of restricted semantics we set $C_W = C$, but in the case of modern semantics we want $C \setminus C_W$ to be infinite as well. For each q_j in $\mathcal{S}$ choose a distinct c_j from C_W and let $\mathcal{S}^\star$ be the system of equations

$$\mathcal{S}^\star \;=\; \{p_1 = 0,\ \ldots,\ p_k = 0\} \,\cup\, \{c_1 q_1^2 = c_1,\ \ldots,\ c_m q_m^2 = c_m\}$$

When working with Boolean algebra or Boolean rings one can replace q_j^2 by q_j.

The first claim is that under either semantics an equation $p = 0$ is a consequence of $\mathcal{S}$ iff one can give an equational proof from the systems $\vdash_{\mathcal{AB}^\star}$, etc., where the $\star$ means in the restricted case to add the rule of inference

$$\frac{c \;=\; 0}{1 \;=\; 0} \qquad \frac{c \;=\; 1}{1 \;=\; 0} \qquad \text{for any } c \in C$$

and in the modern semantics it means that we add the inferences

$$\frac{c \;=\; 0}{1 \;=\; 0} \qquad \text{for any } c \in C_W$$

The second claim is that under either semantics an equation $q \neq 0$ is a consequence of $\mathcal{S}$ iff one can give an equational proof of $cq^2 = c$ from the systems $\vdash_{\mathcal{AB}^\star}$, etc., for some $c \in C_W$.

Bibliography

[1] George Boole, **The Mathematical Analysis of Logic,** Being an Essay Towards a Calculus of Deductive Reasoning. Originally published in Cambridge by Macmillan, Barclay, & Macmillan, 1847. Reprinted in Oxford by Basil Blackwell, 1951.

[2] George Boole, An Investigation of **The Laws of Thought** on Which are Founded the Mathematical Theories of Logic and Probabilities. Originally published by Macmillan, London, 1854. Reprint by Dover, 1958.

[3] S. Burris and H.P. Sankappanavar, **A Course in Universal Algebra.** Springer Verlag, 1981.

[4] Augustus De Morgan, **Formal Logic:** or, the Calculus of Inference, Necessary and Probable. Originally published in London by Taylor and Walton, 1847. Reprinted in London by The Open Court Company, 1926.

[5] Augustus De Morgan, **On the Syllogism,** and Other Logical Writings, edited by Peter Heath, Yale University Press, 1966. (A posthumous collection of De Morgan's papers on logic.)

[6] Theodore Hailperin, **Boole's Logic and Probability.** Studies in Logic and the Foundations of Mathematics, **85**, Elsevier, North-Holland, Amsterdam, New York, Oxford 1976

[7] W. Stanley Jevons, **The Substitution of Similars,** the True Principle of Reasoning, Derived from a Modification of Aristotle's Dictum. Macmillan and Co., London, 1869.

[8] W. Stanley Jevons, **Elementary Lessons in Logic, Deductive and Inductive.** Originally published by Macmillan & Co., London, 1870. Reprinted 1957.

[9] W. Stanley Jevons, **The Principles of Science,** A Treatise on Logic and the Scientific Method. Originally published by Macmillan and Co., London and New York, 1874. Reprinted 1892.

[10] W. Stanley Jevons, **Studies in Deductive Logic,** A Manual for Students. Macmillan and Co. London and New York, 1880.

[11] W. Stanley Jevons, **The Elements of Logic.** Sheldon & Co. New York and Chicago, 1883.

[12] W. Stanley Jevons, **Pure Logic and Other Minor Works** ed. by Robert Adamson and Harriet A. Jevons Lennox Hill Pub. & Dist. Co., NY 1890. Reprinted 1971.

[13] M. Kneale and W. Kneale, **The Development of Logic.** Oxford University Press, 1962.

[14] C.I. Lewis, **A Survey of Symbolic Logic.** Originally published by the Univeristy of Californial Press, 1918. Republished by Dover, 1960.

[15] C.S. Peirce, **Description of a notation for the logic of relatives***, resulting from an amplification of the conceptions of Boole's calculus of logic.* Memoirs of the Amer. Acad. **9** (1870), 317–378. Reprinted in Vol. III of Collected Papers, 27–98.

[16] C.S. Peirce, **On the algebra of logic**. *Chapter I: Syllogistic. Chapter II: The logic of non-relative terms. Chapter III: The logic of relatives.* Amer. Journal of Math. **3** (1880), 15–57. Reprinted in Vol. III of Collected Papers, 104–157.

[17] R.S. Pierce, **Modules over Commutative Regular Rings.** Memoirs AMS No. **70**, 1967.

[18] Ernst Schröder, **Algebra der Logik, Vols. I–III**. 1890–1910, Chelsea reprint 1966.

[19] Richard Whately, **Elements of Logic.** 2nd edition, published in 1826 by J. Mawman, London. Reprinted, with a new introduction, Scholars Facsimiles & Reprints, Inc. 1975.