Substitution/Replacement

In each of the following inferences you are to choose the best answer for how the inference could be accomplished. The four choices are: **substitution**, **replacement**, **both**, **neither**.

1. $$\frac{x+y \approx y+x}{u+v \approx v+u}$$ *Substitution*

2. $$\frac{x\cdot(y+z) \approx (x\cdot y)+(x\cdot z)}{(x+y)\cdot z \approx (x\cdot z)+(y\cdot z)}$$ *Neither*

3. $$\frac{x+y \approx y+x}{(x+y)+z \approx (y+x)+z}$$ *Replacement*

4. $$\frac{x\cdot y \approx u\cdot v}{(x\cdot y)\cdot w \approx (u\cdot v)\cdot w}$$ *Both*

5. $$\frac{x\cdot x \approx x}{(x\cdot x)\cdot(x\cdot x) \approx x\cdot x}$$ *Substitution*

Find a two element **counterexample** to the following equational argument:

$$\frac{x\cdot y \approx x \qquad (x+y)\cdot z \approx (x\cdot z)+(x\cdot y)}{x\cdot(y+z) \approx (x\cdot y)+(x\cdot z)}$$

$+$	a	b
a	u	u
b	v	v

$\cdot$	a	b
a	a	a
b	b	b

$u, v \in \{a, b\}$

$u \neq a$ *or* $v \neq b$

Thus there are three distinct two-element counterexamples.

Find a three element **counterexample** to the following argument:

$$\frac{f^3(x) \approx x}{f^2(x) \approx x}$$

	f
a	b
b	c
c	a

Fill in the reasons for the steps in the following derivation of $(y \cdot x) + (x \cdot y) \approx 0$ from the ring axioms R1–R9 plus the idempotent law $x \cdot x \approx x$:

R1.	$x + 0$	$\approx$	x	given
R2.	$x + (-x)$	$\approx$	0	given
R3.	$x + y$	$\approx$	$y + x$	given
R4.	$x + (y + z)$	$\approx$	$(x + y) + z$	given
R5.	$x \cdot 1$	$\approx$	x	given
R6.	$1 \cdot x$	$\approx$	x	given
R7.	$x \cdot (y \cdot z)$	$\approx$	$(x \cdot y) \cdot z$	given
R8.	$x \cdot (y + z)$	$\approx$	$(x \cdot y) + (x \cdot z)$	given
R9.	$(x + y) \cdot z$	$\approx$	$(x \cdot z) + (y \cdot z)$	given
10.	$x \cdot x$	$\approx$	x	given
11.	$(x + y) \cdot (x + y)$	$\approx$	$x + y$	10 Subs
12.	$(x + y) \cdot (x + y)$	$\approx$	$((x + y) \cdot x) + ((x + y) \cdot y)$	R8 Subs
13.	$(x + y) \cdot x$	$\approx$	$(x \cdot x) + (y \cdot x)$	R9 Subs
14.	$(x \cdot x) + (y \cdot x)$	$\approx$	$x + (y \cdot x)$	10 Repl
15.	$(x + y) \cdot x$	$\approx$	$x + (y \cdot x)$	13, 14 Trans
16.	$((x + y) \cdot x) + ((x + y) \cdot y)$	$\approx$	$(x + (y \cdot x)) + ((x + y) \cdot y)$	15 Repl
17.	$(x + y) \cdot y$	$\approx$	$(x \cdot y) + (y \cdot y)$	R9 Subs
18.	$y \cdot y$	$\approx$	y	10 Subs
19.	$(x \cdot y) + (y \cdot y)$	$\approx$	$(x \cdot y) + y$	18 Repl
20.	$(x + y) \cdot y$	$\approx$	$(x \cdot y) + y$	17, 19 Trans
21.	$(x + (y \cdot x)) + ((x + y) \cdot y)$	$\approx$	$(x + (y \cdot x)) + ((x \cdot y) + y)$	20 Repl
22.	$((x + y) \cdot x) + ((x + y) \cdot y)$	$\approx$	$(x + (y \cdot x)) + ((x \cdot y) + y)$	16, 21 Trans
23.	$(x + y) \cdot (x + y)$	$\approx$	$(x + (y \cdot x)) + ((x \cdot y) + y)$	12, 22 Trans
24.	$(x + (y \cdot x)) + ((x \cdot y) + y)$	$\approx$	$(x + y) \cdot (x + y)$	23 Symm
25.	$(x + (y \cdot x)) + ((x \cdot y) + y)$	$\approx$	$x + y$	11, 24 Trans
26.	$(-x) + (x + (y \cdot x)) + ((x \cdot y) + y)$	$\approx$	$(-x) + (x + y)$	25 Repl
27.	$(-x) + x$	$\approx$	$x + (-x)$	R3 Subs
28.	$(-x) + x$	$\approx$	0	R2, 27 Trans
29.	$(-x) + (x + y)$	$\approx$	$((-x) + x) + y$	R4 Subs

30.	$((-x)+x)+y$	$\approx$	$0+y$	28 Repl
31.	$0+y$	$\approx$	$y+0$	R3 Subs
32.	$y+0$	$\approx$	y	R1 Subs
33.	$0+y$	$\approx$	y	31, 32 Trans
34.	$((-x)+x)+y$	$\approx$	y	30, 33 Trans
35.	$(-x)+(x+y)$	$\approx$	y	29, 34 Trans
36.	$(-x)+\Big((x+(y\cdot x))+((x\cdot y)+y)\Big)$	$\approx$	y	26, 35 Trans
37.	$(-x)+\Big((x+(y\cdot x))+((x\cdot y)+y)\Big)$	$\approx$	$\Big((-x)+(x+(y\cdot x))\Big)+((x\cdot y)+y)$	R4 Subs
38.	$\Big((-x)+(x+(y\cdot x))\Big)+((x\cdot y)+y)$	$\approx$	$(-x)+\Big((x+(y\cdot x))+((x\cdot y)+y)\Big)$	37 Symm
39.	$(-x)+(x+(y\cdot x))$	$\approx$	$((-x)+x)+(y\cdot x)$	R4 Subs
40.	$((-x)+x)+(y\cdot x)$	$\approx$	$0+(y\cdot x)$	28 Repl
41.	$0+(y\cdot x)$	$\approx$	$y\cdot x$	33 Subs
42.	$((-x)+x)+(y\cdot x)$	$\approx$	$y\cdot x$	40, 41 Trans
43.	$(-x)+(x+(y\cdot x))$	$\approx$	$y\cdot x$	39, 42 Trans
44.	$\Big((-x)+(x+(y\cdot x))\Big)+((x\cdot y)+y)$	$\approx$	$(y\cdot x)+((x\cdot y)+y)$	43 Repl
45.	$(y\cdot x)+((x\cdot y)+y)$	$\approx$	$\Big((-x)+(x+(y\cdot x))\Big)+((x\cdot y)+y)$	44 Symm
46.	$(y\cdot x)+((x\cdot y)+y)$	$\approx$	$(-x)+\Big((x+(y\cdot x))+((x\cdot y)+y)\Big)$	38, 45 Trans
47.	$(y\cdot x)+((x\cdot y)+y)$	$\approx$	y	36, 46 Trans
48.	$\Big((y\cdot x)+((x\cdot y)+y)\Big)+(-y)$	$\approx$	$y+(-y)$	47 Repl
49.	$y+(-y)$	$\approx$	0	28 Subs
50.	$\Big(((y\cdot x)+(x\cdot y))+y\Big)+(-y)$	$\approx$	0	48, 49 Trans
51.	$((y\cdot x)+(x\cdot y))+(y+(-y))$	$\approx$	$\Big(((y\cdot x)+(x\cdot y))+y\Big)+(-y)$	R4 Subs
52.	$((y\cdot x)+(x\cdot y))+(y+(-y))$	$\approx$	0	50, 51 Trans
53.	$((y\cdot x)+(x\cdot y))+(y+(-y))$	$\approx$	$((y\cdot x)+(x\cdot y))+0$	49 Repl
54.	$((y\cdot x)+(x\cdot y))+0$	$\approx$	$(y\cdot x)+(x\cdot y)$	R1 Subs
55.	$((y\cdot x)+(x\cdot y))+(y+(-y))$	$\approx$	$(y\cdot x)+(x\cdot y)$	53, 54 Trans
56.	$(y\cdot x)+(x\cdot y)$	$\approx$	$((y\cdot x)+(x\cdot y))+(y+(-y))$	55 Symm
57.	$(y\cdot x)+(x\cdot y)$	$\approx$	0	52, 56 Trans