Give the number of the **Fundamental Equivalence** used for each step below. When justifying item **n** you may use the fundamental equivalences on page 44 as well any of those on the extended list list below whose number is less than **n** (e.g., do not use item 38 to justify item 33):

**32** $\neg 0 \sim 1$

$$\begin{aligned} \neg 0 &\sim \neg 0 \vee 0 && \underline{16} \\ &\sim 0 \vee \neg 0 && \underline{3} \\ &\sim 1 && \underline{11} \end{aligned}$$

**33** $\neg 1 \sim 0$

$$\begin{aligned} \neg 1 &\sim \neg\neg 0 && \underline{32} \\ &\sim 0 && \underline{13} \end{aligned}$$

**34** $P \to 0 \sim \neg P$

$$\begin{aligned} P \to 0 &\sim \neg P \vee 0 && \underline{20} \\ &\sim \neg P && \underline{16} \end{aligned}$$

**35** $P \to 1 \sim 1$

$$\begin{aligned} P \to 1 &\sim \neg P \vee 1 && \underline{20} \\ &\sim 1 && \underline{14} \end{aligned}$$

**36** $0 \to P \sim 1$

$$\begin{aligned} 0 \to P &\sim \neg 0 \vee P && \underline{20} \\ &\sim 1 \vee P && \underline{32} \\ &\sim P \vee 1 && \underline{3} \\ &\sim 1 && \underline{14} \end{aligned}$$

**37** $1 \to P \sim P$

$$\begin{aligned} 1 \to P &\sim \neg 1 \vee P && \underline{20} \\ &\sim 0 \vee P && \underline{33} \\ &\sim P \vee 0 && \underline{3} \\ &\sim P && \underline{16} \end{aligned}$$

**38** $P \to P \sim 1$

$$\begin{aligned} P \to P &\sim \neg P \vee P && \underline{20} \\ &\sim P \vee \neg P && \underline{3} \\ &\sim 1 && \underline{11} \end{aligned}$$

**39** $P \leftrightarrow 0 \sim \neg P$

$$\begin{aligned} P \leftrightarrow 0 &\sim (P \to 0) \wedge (0 \to P) && \underline{29} \\ &\sim \neg P \wedge (0 \to P) && \underline{34} \\ &\sim \neg P \wedge 1 && \underline{36} \\ &\sim \neg P && \underline{15} \end{aligned}$$

**40** $0 \leftrightarrow P \sim \neg P$

$$\begin{aligned} 0 \leftrightarrow P &\sim P \leftrightarrow 0 && \underline{25} \\ &\sim \neg P && \underline{39} \end{aligned}$$

**41** $P \leftrightarrow 1 \sim P$

$$\begin{aligned} P \leftrightarrow 1 &\sim (P \to 1) \wedge (1 \to P) && \underline{29} \\ &\sim 1 \wedge (1 \to P) && \underline{35} \\ &\sim 1 \wedge P && \underline{37} \\ &\sim P \wedge 1 && \underline{4} \\ &\sim P && \underline{15} \end{aligned}$$

**42** $1 \leftrightarrow P \sim P$

$$\begin{aligned} 1 \leftrightarrow P &\sim P \leftrightarrow 1 && \underline{25} \\ &\sim P && \underline{41} \end{aligned}$$

**43** $P \leftrightarrow P \sim 1$

$$\begin{aligned} P \leftrightarrow P &\sim (P \to P) \wedge (P \to P) && \underline{29} \\ &\sim 1 \wedge 1 && \underline{38} \\ &\sim 1 && \underline{15} \end{aligned}$$

**Adequate Sets of Connectives**

Circle (or highlight) the formulas among $0, 1, P, \neg P$ that can be represented by a formula $\mathsf{F}(P)$ using the connectives in $\mathcal{C}$:

| Given | | | | |
|---|---|---|---|---|
| $\mathcal{C} = \{\wedge, \vee\}$ | $0$ | $1$ | $\boxed{P}$ | $\neg P$ |
| $\mathcal{C} = \{\wedge, 0\}$ | $\boxed{0}$ | $1$ | $\boxed{P}$ | $\neg P$ |
| $\mathcal{C} = \{0, \leftrightarrow\}$ | $\boxed{0}$ | $\boxed{1}$ | $\boxed{P}$ | $\boxed{\neg P}$ |
| $\mathcal{C} = \{\neg, \leftrightarrow\}$ | $\boxed{0}$ | $\boxed{1}$ | $\boxed{P}$ | $\boxed{\neg P}$ |
| $\mathcal{C} = \{\rightarrow, 1\}$ | $0$ | $\boxed{1}$ | $\boxed{P}$ | $\neg P$ |

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Circle (or highlight) the connectives that can be realized using the connectives in $\mathcal{C}$:

| Given | | | | |
|---|---|---|---|---|
| $\mathcal{C} = \{\wedge, \vee\}$ | $\boxed{\vee}$ | $\boxed{\wedge}$ | $\rightarrow$ | $\leftrightarrow$ |
| $\mathcal{C} = \{\wedge, 0\}$ | $\vee$ | $\boxed{\wedge}$ | $\rightarrow$ | $\leftrightarrow$ |
| $\mathcal{C} = \{0, \leftrightarrow\}$ | $\vee$ | $\wedge$ | $\rightarrow$ | $\boxed{\leftrightarrow}$ |
| $\mathcal{C} = \{\neg, \leftrightarrow\}$ | $\vee$ | $\wedge$ | $\rightarrow$ | $\boxed{\leftrightarrow}$ |
| $\mathcal{C} = \{\rightarrow, 1\}$ | $\boxed{\vee}$ | $\wedge$ | $\boxed{\rightarrow}$ | $\leftrightarrow$ |

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**Substitution/Replacement**

In each of the following inferences you are to choose the best answer for how the inference could be accomplished. The four choices are: **substitution**, **replacement**, **both**, **neither**.

1. $$\frac{P \sim Q}{P \wedge P \sim P \wedge Q}$$ both

2. $$\frac{P \sim Q}{Q \sim P}$$ substitution

3. $$\frac{P \rightarrow Q \sim \neg P \vee Q}{\neg P \vee Q \sim P \rightarrow Q}$$ neither

4. $$\frac{P \rightarrow Q \sim \neg P \vee Q}{Q \rightarrow (P \rightarrow Q) \sim Q \rightarrow (\neg P \vee Q)}$$ replacement

5. $$\frac{P \rightarrow Q \sim \neg P \vee Q}{Q \rightarrow P \sim \neg Q \vee P}$$ substitution

**Exercise 2.5.8** (page 57 in LMCS):
Prove (by induction) that $\{\vee, \wedge, \rightarrow, \leftrightarrow, 1\}$ is not adequate. Conclude that any adequate set of standard connectives must have either $\neg$ or 0 in it.

**Proof.** We claim that $(\star)$ every formula $\mathsf{F}(P)$ that one can make with the connectives above is equivalent to either 1 or $P$. This means that $\neg P$ cannot be expressed using these connectives, so they do not form an adequate set of connectives.

Then, as the only standard connectives missing from the above collection are $\neg$ and 0, Any set of standard connectives that excludes these two connectives will be a subset of the above set of 5 connectives, and will thus not be adequate.

We prove $(\star)$ by induction. First we give a proof like those in the textbook, and then a proof like that in the transparencies.

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**First Proof:**
First we look at the **Ground Cases**:

- For $\mathsf{F}(P)$ a variable we must have $\mathsf{F}(P) = P$, and then $F$ is equivalent to $P$.
- For $\mathsf{F}(P)$ a constant we must have $\mathsf{F}(P) = 1$, and then $F$ is equivalent to 1.

Then we make the **Induction Hypothesis**: Suppose $\mathsf{G}(P)$ and $\mathsf{H}(P)$ are each equivalent to one of $P$ and 1.

Now we are ready for the proof of the **Induction Step**, namely we want to show that $\mathsf{G}(P) \square \mathsf{H}(P)$ is equivalent to either $P$ or 1 for $\square$ one of the given (in this case, standard) binary connectives.

We know that $\mathsf{G}(P) \square \mathsf{H}(P)$ is equivalent to one of $P \square P$, $P \square 1$, $1 \square P$, and $1 \square 1$. So it suffices to show that each of these four cases is equivalent to one of $P$ or 1 when $\square$ is chosen from the given set of connectives. This follows from the following table:

| $P \square P$ | $P \square 1$ | $1 \square P$ | $1 \square 1$ |
|---|---|---|---|
| $P \vee P \sim P$ | $P \vee 1 \sim 1$ | $1 \vee P \sim 1$ | $1 \vee 1 \sim 1$ |
| $P \wedge P \sim P$ | $P \wedge 1 \sim P$ | $1 \wedge P \sim P$ | $1 \wedge 1 \sim 1$ |
| $P \wedge P \rightarrow 1$ | $P \wedge 1 \rightarrow 1$ | $1 \wedge P \rightarrow P$ | $1 \wedge 1 \rightarrow 1$ |
| $P \leftrightarrow P \sim 1$ | $P \leftrightarrow 1 \sim P$ | $1 \leftrightarrow P \sim P$ | $1 \leftrightarrow 1 \sim 1$ |

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**Second Proof:**
Suppose we can find a formula $\mathsf{F}(P)$ using the given connectives that is not equivalent to one of $P$ or 1. Then choose $\mathsf{F}(P)$ to be a smallest possible formula with this property. Clearly $\mathsf{F}(P)$ is not $P$ or 1, so it must look like $\mathsf{G}(P) \square \mathsf{H}(P)$ where $\square$ is one of the given connectives. $\mathsf{G}(P)$ and $\mathsf{H}(P)$ are smaller than $\mathsf{F}(P)$, so they are each equivalent to one of $P$ or 1. But then we can **use the table above** to see that $\mathsf{F}(P)$ is also equivalent to one of $P$ or 1. Contradiction!

**BONUS** *(Do not expect any help from the instructor or TAs on this one—you are on your own!)*

Show that a ternary connective $\tau$ is adequate iff at least one of $\tau(P,Q,Q)$, $\tau(P,P,Q)$, and $\tau(P,Q,P)$ is equivalent to either $P|Q$ or $P \curlywedge Q$.

**Proof.** The direction ($\Longleftarrow$) is trivial as each of $|$ and $\curlywedge$ is adequate.

For the direction ($\Longrightarrow$) we assume $\tau$ is adequate. Then $\tau(P,P,P) \sim \neg P$, for otherwise we cannot express $\neg P$ using $\tau$. Thus each of $\tau(P,Q,Q)$, $\tau(P,P,Q)$, $\tau(P,Q,P)$ has a truth table that fills out

| $P$ | $Q$ | |
|---|---|---|
| 1 | 1 | 0 |
| 1 | 0 | |
| 0 | 1 | |
| 0 | 0 | 1 |

so each of the three is equivalent to one of $\neg P$, $\neg Q$, $P|Q$, $P \curlywedge Q$.

Suppose none of the three is equivalent to either of $P|Q$ or $P \curlywedge Q$. Then each of the three is equivalent to one of $\neg P$ or $\neg Q$. But then it is easy to argue that each of the eight formulas

$$\tau(P,P,P),\ \tau(P,P,\neg P),\ \ldots\ ,\ \tau(\neg P,\neg P,\neg P)$$

is equivalent to either $P$ or $\neg P$. This means that no $\mathsf{F}(P)$ using only the connective $\tau$ can be equivalent to 0 (or to 1), i.e., the constant 0 is not expressible using $\tau$. This contradicts our assumption that $\tau$ is adequate. Thus at least one of $\tau(P,Q,Q)$, $\tau(P,P,Q)$, $\tau(P,Q,P)$ is equivalent to either $P|Q$ or $P \curlywedge Q$.