The Dedekind–Peano Number System

Let P be the set of positive natural numbers.
Let $'$ be the successor function.

PEANO's AXIOMS

P1: 1 is not the successor of any number.

P2: If $m' = n'$, then $m = n$.

P3: (*Induction*) If $X \subseteq P$ is closed under successor, and if $1 \in X$, then $X = P$.

Definition B.0.1 [Addition] Let addition be defined as follows:

i. $n + 1 \;=\; n'$

ii. $m + n' \;=\; (m + n)'$

Lemma B.0.2 $m' + n \;=\; m + n'$

Proof: (By induction on n.)

For $n = 1$:

$m' + 1$	$=$	$(m')'$	by	B.0.1 i
	$=$	$(m + 1)'$	by	B.0.1 i
	$=$	$m + 1'$	by	B.0.1 ii

Induction Hypothesis: $m' + n \; = \; m + n'$

Proof of Induction Step:

$$
\begin{aligned}
m' + n' &= (m' + n)' && \text{by } \boxed{\text{B.0.1 ii}} \\
&= (m + n')' && \text{by } \boxed{\text{Ind Hyp}} \\
&= m + n'' && \text{by } \boxed{\text{B.0.ii}}
\end{aligned}
$$

Lemma B.0.3 $m' + n \; = \; (m + n)'$

Proof:

$$
\begin{aligned}
m' + n &= m + n' && \text{by } \boxed{\text{B.0.2}} \\
&= (m + n)' && \text{by } \boxed{\text{B.0.1 ii}}
\end{aligned}
$$

Lemma B.0.4 $1 + n = n'$

Proof: (By induction on n.)

For $n = 1$:

$1 + 1 = 1'$ by B.0.1 i

Induction Hypothesis: $1 + n = n'$

Proof of Induction Step:

$$\begin{aligned} 1 + n' &= (1 + n)' && \text{by } \boxed{\text{B.0.1 ii}} \\ &= n'' && \text{by } \boxed{\text{Ind Hyp}} \end{aligned}$$

Lemma B.0.5 $1 + n = n + 1$

Proof:

$$\begin{aligned} 1 + n &= n' && \text{by } \boxed{\text{B.0.4}} \\ &= n + 1 && \text{by } \boxed{\text{B.0.1 i}} \end{aligned}$$

Lemma B.0.6 $m + n = n + m$

Proof: (By induction on n.)

For $n = 1$:

$m + 1 = 1 + m$ by B.0.5

Induction Hypothesis: $m + n = n + m$

Proof of Induction Step:

$m + n'$	$=$	$(m + n)'$	by	B.0.1 ii
	$=$	$(n + m)'$	by	Ind Hyp
	$=$	$n + m'$	by	B.0.1 ii
	$=$	$n' + m$	by	B.0.2

Definition B.0.8 [Multiplication]
Let multiplication be defined as follows:

i. $n \cdot 1 = n$
ii. $m \cdot n' = (m \cdot n) + m$

Definition B.0.15 [Exponentiation]
Let exponentiation be defined as follows:

i. $a^1 = a$
ii. $a^{n'} = a^n \cdot a$