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# PMATH 330 Final Exam, Winter 1998

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NO AIDS PERMITTED.

There are 12 pages in this exam.

Make sure you have all 12 pages.

| Problem | (out of) | Marks |
|---|---|---|
| 1 | (10) | |
| 2 | (10) | |
| 3 | (10) | |
| 4 | (10) | |
| 5 | (10) | |
| 6 | (10) | |
| 7 | (10) | |
| 8 | (10) | |
| 9 | (10) | |
| 10 | (10) | |
| TOTAL | (100) | |

Optional Assignment Bonus ______

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**Problem 1.**

**(a)** Write out first-order sentences which express the following statements about natural numbers: (you may use the abbreviations $prime(x)$ for a formula expressing '$x$ is prime', $x|y$ for a formula expressing '$x$ divides $y$'.)

- Every natural number is the sum of four squares.

- Any two positive integers have a least common multiple.

- There are an infinite number of primes of the form $x^2 + 1$.

**(b)** Determine, by any means you know, whether the following two formulas are truth equivalent. Be sure to give your reasons.

$$\begin{array}{lll} F(P,Q,R,S,T,U) & \text{is} & P \to (Q \to (R \to (S \to (T \to U)))) \\ G(P,Q,R,S,T,U) & \text{is} & (P \wedge Q \wedge R \wedge S \wedge T) \to U \end{array}$$

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**Problem 2.**
Suppose you encounter three members **A**, **B**, **C** of the island of Tufa (remember that the Tu's always tell the truth, the Fa's always lie). They each give you a statement which we will assume you have translated into propositional logic as follows, where $A$ denotes the statement: " **A** is a Tu", etc:

- **A** says: $\neg(B \vee C \rightarrow \neg(\neg A \leftrightarrow B \wedge C))$
- **B** says: $(A \leftrightarrow C) \leftrightarrow (B \leftrightarrow (B \leftrightarrow C))$
- **C** says: $(A \vee B \vee C) \wedge (\neg A \vee \neg B \vee \neg C)$

From this information determine, as best possible, the tribe to which each of the three belongs.

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**Problem 3.**
You are given the following information about four minerals $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ that are found in ore samples at the Bre-X site in Indonesia:

**(a)** The case that either **A** is missing or **B** appears holds if and only if either **C** or **D** is missing.

**(b)** The case that either **B** or **C** appears holds if and only if either **A** or **D** is missing.

**(c)** The case that either **C** or **D** appears holds if and only if either **A** or **B** is missing.

Determine from this information, using the method of Boole, the most general conclusion that you can draw concerning only the two minerals **B** and **C**. And be sure to express the final conclusion in English!

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**Problem 4.**
Transform the following argument in first-order logic into a set of clauses, such that the set of clauses will be unsatisfiable if and only if the argument is valid. You do NOT need to test the argument for validity.

$\forall x \exists y \, ((x < y) \vee (y < x))$
$\forall x \forall y \, ((x < y) \rightarrow \exists z \, ((x < z) \wedge (z < y)))$
Therefore $\exists x \forall y \, (\neg \, (x \approx y) \rightarrow (x < y))$

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**Problem 5.**

You are given the one-rule TRS $\mathcal{R} = \{fgfgx \longrightarrow fggx\}$, where $f, g$ are unary.

**(a)** Why is $\mathcal{R}$ terminating?

**(b)** Find all critical pairs of $\mathcal{R}$.

**(c)** Apply the Critical Pairs Lemma to determine if $\mathcal{R}$ is a normal form TRS.

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**Problem 6.**
Fill in the reasons (Birkshoff's Rules) for the following derivation of $(x \cdot y) \cdot (y \cdot x) \approx 1$ from the set of equations defining idempotent monoids:

| | *EQUATION* | *REASON* |
|---|---|---|
| 1. | $(x \cdot y) \cdot z \approx x \cdot (y \cdot z)$ | *given* |
| 2. | $x \cdot 1 \approx x$ | *given* |
| 3. | $1 \cdot x \approx x$ | *given* |
| 4. | $x \cdot x \approx 1$ | *given* |
| 5. | $y \cdot y \approx 1$ | |
| 6. | $x \cdot (y \cdot y) \approx x \cdot 1$ | |
| 7. | $x \cdot (y \cdot y) \approx x$ | |
| 8. | $(x \cdot y) \cdot y \approx x \cdot (y \cdot y)$ | |
| 9. | $(x \cdot y) \cdot y \approx x$ | |
| 10. | $((x \cdot y) \cdot y) \cdot x \approx x \cdot x$ | |
| 11. | $((x \cdot y) \cdot y) \cdot x \approx (x \cdot y) \cdot (y \cdot x)$ | |
| 12. | $(x \cdot y) \cdot (y \cdot x) \approx ((x \cdot y) \cdot y) \cdot x$ | |
| 13. | $(x \cdot y) \cdot (y \cdot x) \approx x \cdot x$ | |
| 14. | $(x \cdot y) \cdot (y \cdot x) \approx 1$ | |

**Problem 7**

**(a)** Fill in the Cayley table

| $\cdot$ | $a$ | $b$ |
|---|---|---|
| $a$ | | |
| $b$ | | |

to define a two-element algebraic structure $\mathbf{A} = (\{a, b, c\}, \cdot)$ that provides a counterexample to the following argument:

$$\begin{array}{rcl} x \cdot x & \approx & x \\ (x \cdot y) \cdot z & \approx & x \cdot (y \cdot z) \\ \therefore x \cdot y & \approx & y \cdot x \end{array}$$

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**(b)** Show that the following sentence is not valid (where $R, S, T$ are unary predicates):

$$\forall x \exists y \forall z \big( (Rx \to (Sy \to Tz)) \to ((Rx \to Sy) \to Tz) \big)$$

**Problem 8.**

Consider the following 5 sentences in the language of directed graphs:

$\mathsf{F}_1$: $\forall x\,(rxx)$
$\mathsf{F}_2$: $\exists x \forall y\,(rxy)$
$\mathsf{F}_3$: $\forall x \forall y \forall z\,(rxy \wedge ryz \to rxz)$
$\mathsf{F}_4$: $\forall x \forall y (rxy \to \exists z\,(rxz \wedge rzy))$
$\mathsf{F}_5$: $\forall x \exists y \forall z\,(rxy \wedge (rxz \to ryz))$

and the following two directed graphs:

$\mathbf{G}_1 = (G, r)$ where $G = \{0, 1, 2, 3\}$ and $r = \{(0,1), (1,1), (2,1), (3,1), (3,2)\}$

$\mathbf{G}_2 = (N, |)$ where $N = \{0, 1, 2, \ldots\}$, the non-negative integers, with the usual 'divides' relation.

Find the truth values of each of the above sentences in each of the above structures, and enter these values (0 or 1) in the table below.

[This question will be marked as follows: each correct answer is worth 1 mark; each incorrect answer receives a penalty of -1; each blank receives 0 marks. However, the lowest possible total mark for this question is 0.]

| | $\mathsf{F}_1$ | $\mathsf{F}_2$ | $\mathsf{F}_3$ | $\mathsf{F}_4$ | $\mathsf{F}_5$ |
|---|---|---|---|---|---|
| $\mathbf{G}_1$ | | | | | |
| $\mathbf{G}_2$ | | | | | |

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**Problem 9**

You are given the language $\mathcal{L} = \{r, a, b\}$ where $r$ is a binary relation symbol and $a$, $b$ are constant symbols. For the set $\mathcal{S}$ of clauses (in this language) given by

$$\{rab\} \qquad \{\neg\, rxy,\ \neg\, ryz,\ rxz\} \qquad \{\neg\, rxy,\ ryx\} \qquad \{\neg\, rxx\}$$

**(a)** Show $\mathcal{S}$ is not satisfiable by using resolution on ground instances.

**(b)** Show $\mathcal{S}$ is not satisfiable by using resolution with most general opp-unifiers.

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**Problem 10.**

**(a)** Convert the following sentence (in the language of directed graphs) into an equivalent sentence in prenex form, and then skolemize it:

$$\exists x\,(\exists y\,(\forall x\,(rxy) \vee \exists z\,(ryz)) \to \neg\,\exists y\,(ryx)).$$

**(b)** Let $\mathcal{L}$ be the first-order language which consists of one binary relation symbol $r$ and one unary function symbol $f$, and let $\mathbf{S}$ be the $\mathcal{L}$-structure given by:

$\mathbf{S} = \{a, b\}$

| $r$ | $a$ | $b$ |
|---|---|---|
| $a$ | 1 | 1 |
| $b$ | 0 | 1 |

| | $f$ |
|---|---|
| $a$ | $b$ |
| $b$ | $b$ |

Let $\mathsf{F}(x, y)$ be the formula $\boxed{\exists z\,(fz \approx y \vee \forall z\,((rxz) \wedge (rzy)) \vee (rfzy))}$ Find the relation $\mathsf{F}^{\mathbf{S}}$ which $\mathsf{F}$ defines in $\mathbf{S}$.

**Alternate Problem or Bonus Homework.**
This problem is optional. It may be used to replace one of the first 10, or it may be used as the equivalent of a bonus problem assignment. If you want to use this problem, please circle which of the Problems 1–10 this is to replace, or circle 'Bonus' if you would like it to be a bonus problem assignment.

1 2 3 4 5 6 7 8 9 10 Bonus

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**(a)** State and prove the Compactness Theorem for Propositional Logic.

**(b)** Apply the Compactness Theorem for First-Order Logic to show that there are non-standard models of Th($\mathbf{N}$), the first-order theory of the natural numbers $\mathbf{N} = (N, +, \cdot, <, 0, 1)$.

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