(COVER PAGE)

- Instructor: S. Burris
- Course: PMath 330 (Check your section: ☐ §1 at 10:30 a.m. ☐ §2 at 1:30 p.m.
- No Aids
- Marks

1.	/8
2.	/10
3.	/10
4.	/8
5.	/9
6.	/8
7.	/10
8.	/8
9.	/9
10.	/10
11.	/6
12.	/10

Problem 1

8 marks

You are given the following information about four classes A, B, C and D.

(a) $A \cup B \ \subseteq \ C' \cup D'$

(b) $C \cup D \ \subseteq \ A \cup B$

One can express the two facts above as a single equation $\mathsf{E}(A, B, C, D) = 0$ where:

$\mathsf{E}(A, B, C, D) \ =$ ______________________________

Determine from this information, using the method of Boole, the most general conclusion that you can draw concerning only the two classes C and D.

Problem 2

10 marks

There are two tribes on the island of Tufa — the Tu's, who always tell the truth, and the Fa's, who always lie. A traveller encountered three residents of the island, A, B, and C, and each made a statement to the traveller:

(a) A said " B or C is telling the truth — but not both of them".

(b) B said " Some of us are telling the truth ".

(c) C said "If A is lying then B is telling the truth iff I am lying".

Translate this into **propositional logic**, and use propositional logic to determine as best possible which tribes A, B, and C belong to.

Problem 3

10 marks

Fill in the reasons for the following resolution argument:

1.	$\{P, Q\}$	given	10.	$\{Q, \neg R\}$	
2.	$\{R, S\}$	given	11.	$\{Q, \neg T\}$	
3.	$\{T, U\}$	given	12.	$\{\neg S, \neg T\}$	
4.	$\{\neg P, \neg R\}$	given	13.	$\{\neg R, \neg U\}$	
5.	$\{\neg Q, \neg S\}$	given	14.	$\{S, \neg T\}$	
6.	$\{\neg P, \neg T\}$	given	15.	$\{S, \neg U\}$	
7.	$\{\neg Q, \neg U\}$	given	16.	$\{\neg U\}$	
8.	$\{\neg R, \neg T\}$	given	17.	$\{\neg T\}$	
9.	$\{\neg S, \neg U\}$	given	18.	$\{U\}$	
			19.	$\{\ \}$	

Problem 4

8 marks

Give explict formulas that express the following standard connectives in terms of the Sheffer connective '|' (and propositional variables):

$\neg P \sim$ __________

$P \vee Q \sim$ __________

$P \wedge Q \sim$ __________

$P \to Q \sim$ __________

Problem 5

9 marks

Find a two-element **counterexample** to the following equational argument:

$$\begin{array}{l} x \cdot y \ \approx \ y \\ x + y \ \approx \ y + x \\ \hline \therefore (x+y) \cdot z \ \approx \ (x \cdot z) + (y \cdot z) \end{array}$$

$+$	a	b
a		
b		

$\cdot$	a	b
a		
b		

Find a three-element **counterexample** to the following argument:

$$\begin{array}{l} fffx \ \approx \ ffx \\ \hline \therefore ffx \ \approx \ fx \end{array}$$

	f
a	
b	
c	

Problem 6

8 marks

Fill in the reasons for the steps in the following derivation:

1.	$x \cdot 1$	$\approx$	x	given
2.	$x \cdot x^{-1}$	$\approx$	1	given
3.	$(x \cdot y) \cdot z$	$\approx$	$x \cdot (y \cdot z)$	given
4.	1	$\approx$	$x \cdot x^{-1}$	______
5.	$1 \cdot (x^{-1})^{-1}$	$\approx$	$(x \cdot x^{-1}) \cdot (x^{-1})^{-1}$	______
6.	$(x \cdot x^{-1}) \cdot (x^{-1})^{-1}$	$\approx$	$x \cdot (x^{-1} \cdot (x^{-1})^{-1})$	______
7.	$1 \cdot (x^{-1})^{-1}$	$\approx$	$x \cdot (x^{-1} \cdot (x^{-1})^{-1})$	______
8.	$x^{-1} \cdot (x^{-1})^{-1}$	$\approx$	1	______
9.	$x \cdot (x^{-1} \cdot (x^{-1})^{-1})$	$\approx$	$x \cdot 1$	______
10.	$x \cdot (x^{-1} \cdot (x^{-1})^{-1})$	$\approx$	x	______
11.	$1 \cdot (x^{-1})^{-1}$	$\approx$	x	______

Problem 7

10 marks

You are given the one-rule TRS $\mathcal{R} = \{gfgfu \longrightarrow gffu\}$, where f, g are unary.

(a) Why is $\mathcal{R}$ terminating?

(b) Find all nontrivial critical pairs of $\mathcal{R}$.

(c) Apply the Critical Pairs Lemma to determine if $\mathcal{R}$ is a normal form TRS.

Problem 8

8 marks

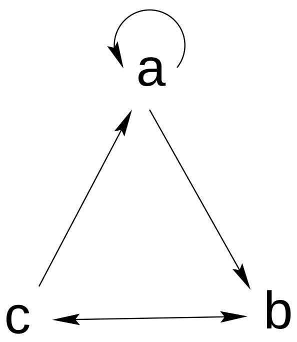
r is a binary relation symbol and f a unary function symbol. Fill in the reasons for the following resolution derivation:

1.	$\{\neg\, rfxx\}$	given
2.	$\{\neg\, rxy,\ \neg\, rxz,\ ryz\}\}$	given
3.	$\{rfffxx\}$	given
4.	$\{\neg\, rxy,\ rfxfy\}$	given
5.	$\{rffxx\}$	given
6.	$\{\neg\, rxfy,\ \neg\, rxy\}$	
7.	$\{\neg\, rfffxfx\}$	
8.	$\{\neg\, rffxx\}$	
8.	$\{\ \}$	

Problem 9

9 marks

Determine the binary relation defined by the formula $F(x, y) = \forall u\,(rxu \to \neg\, ruy)$ on the following directed graph:



	a	b	c
a			
b			
c			

Problem 10

10 marks

Consider the following 5 sentences in the language of directed graphs:

F_1: $\forall x \neg (rxx)$

F_2: $\exists x \forall y (ryx)$

F_3: $\forall x \forall y \forall z (\neg rxy \vee \neg ryz \vee rxz)$

F_4: $\forall x \forall y \big(\neg rxy \vee \exists z (rxz \wedge rzy)\big)$

F_5: $\forall x \exists y \forall z (rxy \to (\neg rxz \vee ryz))$

and the following two directed graphs:

$\mathbf{G}_1 = (G, r)$ where $G = \{0, 1\}$ and $r = \{(0,1), (1,0)\}$

$\mathbf{G}_2 = (N, |)$ where $N = \{0, 1, 2, \ldots\}$, the non-negative integers, with the usual 'divides' relation.

Find the truth values of each of the above sentences in each of the above structures, and enter these values (0 or 1) in the table below.

[This question will be marked as follows: each correct answer is worth 1 mark; each incorrect answer receives a penalty of -1; each blank receives 0 marks. However, the lowest possible total mark for this question is 0.]

	F_1	F_2	F_3	F_4	F_5
$\mathbf{G}_1$					
$\mathbf{G}_2$					

Problem 11

6 marks

- Put the following formula in prenex form:

$$\forall x\,(x < u) \to \big[\exists v \forall w\,(x + v < x + w) \to \forall z\,(z \cdot v < z \cdot w)\big]$$

Answer:

- Skolemize the following formula:

$$\exists x \exists y \forall z \exists w\,\big(x + y < z \;\leftrightarrow\; y < z + w\big)$$

Answer:

Problem 12

10 marks

Convert the following argument into a set $\mathcal{C}$ of clauses such that the argument is valid iff the set $\mathcal{C}$ is not satisfiable.

$$\forall x \Big(\exists y \, (x \cdot y \leq x) \;\to \neg\, \exists y \forall z \, \big((y + z < x \cdot z) \land \neg\, (y \cdot x \leq z) \big) \Big) \qquad \therefore\; \exists x \forall y \, (x + y < x \cdot y)$$

Answer: