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# PMATH 330 Final Exam, Fall 1998

Instructor: S. Burris

NO AIDS PERMITTED.

There are 11 pages in this exam.

Make sure you have all 11 pages.

| Problem | (out of) | Marks |
|---|---|---|
| 1 | (10) | |
| 2 | (10) | |
| 3 | (10) | |
| 4 | (10) | |
| 5 | (10) | |
| 6 | (10) | |
| 7 | (10) | |
| 8 | (10) | |
| 9 | (10) | |
| TOTAL | (90) | |

Optional Bonus (5) ______

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**Problem 1** Consider the following 10 formulas with their truth tables:

1: $\neg(\neg Q \land S)$
2: $\neg(P \lor (Q \lor \neg R)) \lor R$
3: $(S \lor (S \land Q)) \leftrightarrow (Q \land S)$
4: $(P \lor (R \land (Q \leftrightarrow P))) \lor R$
5: $((\neg(P \leftrightarrow R) \to Q) \to P) \leftrightarrow S$
6: $(((R \to Q) \lor (S \land Q)) \lor R) \leftrightarrow Q$
7: $((S \leftrightarrow P) \lor R) \to (Q \lor (Q \lor P))$
8: $(R \lor P) \to (((P \land R) \land S) \leftrightarrow (S \lor Q))$
9: $((P \lor Q) \land ((Q \to (Q \to S)) \lor Q)) \land (R \land \neg P)$
10: $(((R \land S) \land (Q \land (S \to Q))) \to (\neg(Q \lor R) \to R)) \land R$

| | P | Q | R | S | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1. | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 |
| 2. | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 1 |
| 3. | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 |
| 4. | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 0 |
| 5. | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 |
| 6. | 1 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 1 |
| 7. | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 0 |
| 8. | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 |
| 9. | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 |
| 10. | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 |
| 11. | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 0 |
| 12. | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 0 |
| 13. | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 |
| 14. | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 1 |
| 15. | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 |
| 16. | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 |

**(a)** Find the conjunctive normal form of 7.

**(b)** Determine which of the formulas are truth equivalent to 2.

**(c)** Find all the tautologies; all the contradictions.

**(d)** Determine if the set $\{1, 2, 3, 4, 5, 6, 7, 9, 10\}$ of formulas is satisfiable?

**(e)** Determine if the argument $3, 7, 8 \therefore 10$ is valid.

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**Problem 2.**

You are given the following premisses of an argument involving four classes $A, B, C, D$ that has been translated into the calculus of classes:

$$\begin{aligned} (AD' \cup A'D)(B \cup C) &= 0 \\ (A \cup D)B'C'(A' \cup D') &= 0 \end{aligned}$$

Determine from this information, using the method of Boole, the most general conclusion that you can draw concerning only the two classes $A$ and $D$.

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**Problem 3**

**(a)** Use the method of Venn diagrams to determine if the syllogism

$$\begin{array}{l} \text{No P is M} \\ \text{All S is M} \\ \hline \text{Some S is not P} \end{array}$$

is valid. (Be sure to consider the modern versus Aristotelian assumptions.)

**(b)** Let $\mathbf{A} = (A, \vee, \wedge)$ be an algebra with two binary operations on the set $A = \{0, 1\}$ where the operations are defined by

$$\begin{aligned} x \vee y &= \max(x, y) \\ x \wedge y &= \min(x, y) \end{aligned}$$

Determine if $\mathbf{A}$ satisfies the distributive law:

$$x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z)$$

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**Problem 4.**
Fill in the reasons (Birkshoff's Rules) for the following derivation:

| | *EQUATION* | *REASON* |
|---|---|---|
| 1. | $x \vee (x \wedge y) \approx x$ | *given* |
| 2. | $x \wedge (x \vee y) \approx x$ | *given* |
| 3. | $x \vee (y \vee z) \approx (x \vee y) \vee z$ | *given* |
| 4. | $x \wedge y \approx y \wedge x$ | *given* |
| 5. | $x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z)$ | *given* |
| 6. | $x \vee (x \wedge z) \approx x$ | |
| 7. | $(x \vee (x \wedge z)) \vee (y \wedge z) \approx x \vee (y \wedge z)$ | |
| 8. | $x \vee ((x \wedge z) \vee (y \wedge z)) \approx (x \vee (x \wedge z)) \vee (y \wedge z)$ | |
| 9. | $x \vee ((x \wedge z) \vee (y \wedge z)) \approx x \vee (y \wedge z)$ | |
| 10. | $x \vee (y \wedge z) \approx x \vee ((x \wedge z) \vee (y \wedge z))$ | |
| 11. | $x \wedge z \approx z \wedge x$ | |
| 12. | $x \vee ((x \wedge z) \vee (y \wedge z)) \approx x \vee ((z \wedge x) \vee (y \wedge z))$ | |
| 13. | $x \vee (y \wedge z) \approx x \vee ((z \wedge x) \vee (y \wedge z))$ | |
| 14. | $y \wedge z \approx z \wedge y$ | |
| 15. | $x \vee ((z \wedge x) \vee (y \wedge z)) \approx x \vee ((z \wedge x) \vee (z \wedge y))$ | |

**Problem 5.**
Find all critical pairs for the two term rewrite rules $\left\{ \begin{array}{lcl} x + (y \cdot z) & \to & x \cdot z \\ x \cdot (y + z) & \to & x + z \end{array} \right.$

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**Problem 6**

You are given the language $\mathcal{L} = \{P, f, a, b\}$ where $P$ is a unary relation symbol, $f$ is a unary operation symbol, and $a, b$ are a constant symbols. For the set $\mathcal{S}$ of clauses (in this language) given by

$$\{Pa,\ Pb\} \qquad \{\neg\, Px,\ Pfx\} \qquad \{\neg\, Pfa\} \qquad \{\neg\, Pffb\}$$

**(a)** Show $\mathcal{S}$ is not satisfiable by using resolution on ground instances.

**(b)** Show $\mathcal{S}$ is not satisfiable by using resolution with most general opp-unifiers.

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**Problem 7.**

**(a)** Write out first-order sentences using $\mathcal{L} = \{+, \cdot, <, 0, 1\}$ which express the following statements about **natural numbers**. You may use the abbreviations $prime(x)$ for a formula expressing '$x$ is prime', $x|y$ for a formula expressing '$x$ divides $y$'.

- The sum of two odd primes is even.

- Any two positive numbers have a greatest common divisor.

- There are an infinitely many solutions to $x^2 - y^2 = 1$.

**(b)** Determine, by any means you know, whether the following two formulas are truth equivalent. Be sure to give your reasons.

$$\begin{array}{lll} F(P,Q,R,S,T,U) & \text{is} & P \leftrightarrow (Q \leftrightarrow (R \leftrightarrow (S \leftrightarrow (T \leftrightarrow U)))) \\ G(P,Q,R,S,T,U) & \text{is} & \neg P \leftrightarrow (Q \leftrightarrow (R \leftrightarrow (S \leftrightarrow (T \leftrightarrow \neg U)))) \end{array}$$

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**Problem 8.**

Consider the following 5 sentences in the language of directed graphs:

$\mathsf{F}_1$: $\forall x\, (\neg\, rxx)$

$\mathsf{F}_2$: $\forall x \exists y\, (\neg\, rxy)$

$\mathsf{F}_3$: $\forall x \forall y \forall z\, (rxy \wedge ryz \rightarrow rxz)$

$\mathsf{F}_4$: $\exists x \forall y (rxy \rightarrow \exists z\, (rxz \vee rzy))$

$\mathsf{F}_5$: $\forall x \exists y \forall z\, (rxy \rightarrow (rxz \wedge ryz))$

and the following two directed graphs:

$\mathbf{G}_1 = (G, r)$ where $G = \{a, b, c\}$ and $r = \{(a, b), (b, b), (c, a)\}$

$\mathbf{G}_2 = (N, >)$ where $N = \{0, 1, 2, \ldots\}$, the non-negative integers, with the usual 'greater than' relation.

Find the truth values of each of the above sentences in each of the above structures, and enter these values (0 or 1) in the table below.

[This question will be marked as follows: each correct answer is worth 1 mark; each incorrect answer receives a penalty of -1; each blank receives 0 marks. However, the lowest possible total mark for this question is 0.]

| | $\mathsf{F}_1$ | $\mathsf{F}_2$ | $\mathsf{F}_3$ | $\mathsf{F}_4$ | $\mathsf{F}_5$ |
|---|---|---|---|---|---|
| $\mathbf{G}_1$ | | | | | |
| $\mathbf{G}_2$ | | | | | |

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**Problem 9.**
Transform the following argument in first-order logic into a set of clauses, such that the set of clauses will be unsatisfiable if and only if the argument is valid. You do NOT need to test the argument for validity.

$\forall x \exists y\, ((x|y) \rightarrow (Py \rightarrow Px))$
$\forall x \forall y\, ((Px \wedge Py) \rightarrow \exists z\, (Pz))$
Therefore $\neg \exists x \forall y\, (\neg\, (x|y) \rightarrow (Px \leftrightarrow Py))$

**Alternate Problem or Bonus**
This problem is optional. It may be used to replace one of the first 9 (worth 10 points), or it may be used as a Bonus problem (worth 5 points). If you want to use this problem, please circle which of the Problems 1–9 this is to replace, or circle 'Bonus' if you would like it to be a bonus problem.

1 2 3 4 5 6 7 8 9 Bonus

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**Problem**
Prove by **induction on formulas** that the set of connectives $\{\wedge, \vee, \rightarrow\}$ is not adequate. (Note: do not give a proof by induction on the *size* of formulas.)

**Proof**

Assert($F$) **is:**

**Ground Step:**

**Induction Step:** (be sure to state any Induction Hypothesis used)

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