

### Substitution/Replacement

In each of the following inferences you are to choose the answer that gives the most ways in which the inference can be accomplished. The four choices are: **substitution**, **replacement**, **both**, **neither**.

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|----|---|---|
| 1. | $\frac{x + x \approx 0}{(x + x) + (x + x) \approx 0}$   | <div style="border-top: 1px solid black; display: inline-block; padding-top: 5px;">Substitution</div> |
| 2. | $\frac{x + (y + z) \approx (x + y) + z}{(x + y) + z \approx x + (y + z)}$                                 | <div style="border-top: 1px solid black; display: inline-block; padding-top: 5px;">Neither</div>      |
| 3. | $\frac{x \cdot y \approx x \cdot z}{(y \cdot z) \cdot (x \cdot y) \approx (y \cdot z) \cdot (x \cdot z)}$ | <div style="border-top: 1px solid black; display: inline-block; padding-top: 5px;">Both</div>         |
| 4. | $\frac{x \cdot y \approx y \cdot x}{y \cdot (x \cdot y) \approx y \cdot (y \cdot x)}$                     | <div style="border-top: 1px solid black; display: inline-block; padding-top: 5px;">Replacement</div>  |
| 5. | $\frac{x + y \approx u + v}{y + (x + y) \approx y + (u + v)}$   | <div style="border-top: 1px solid black; display: inline-block; padding-top: 5px;">Both</div>         |

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Find a two element **counterexample** to the following equational argument:

$\frac{x \cdot y \approx u \cdot v}{x \cdot (y + z) \approx (x \cdot y) + (x \cdot z)}$	<table style="border-collapse: collapse; margin: auto;"> <tr> <td style="border-right: 1px solid black; padding: 5px 10px;"><math>+</math></td> <td style="padding: 5px 10px;"><math>a</math></td> <td style="padding: 5px 10px;"><math>b</math></td> </tr> <tr style="border-top: 1px solid black;"> <td style="border-right: 1px solid black; padding: 5px 10px;"><math>a</math></td> <td style="padding: 5px 10px;"><math>b</math></td> <td style="padding: 5px 10px;"><math>\star</math></td> </tr> <tr> <td style="border-right: 1px solid black; padding: 5px 10px;"><math>b</math></td> <td style="padding: 5px 10px;"><math>\star</math></td> <td style="padding: 5px 10px;"><math>\star</math></td> </tr> </table>	$+$	$a$	$b$	$a$	$b$	$\star$	$b$	$\star$	$\star$	<table style="border-collapse: collapse; margin: auto;"> <tr> <td style="border-right: 1px solid black; padding: 5px 10px;"><math>\cdot</math></td> <td style="padding: 5px 10px;"><math>a</math></td> <td style="padding: 5px 10px;"><math>b</math></td> </tr> <tr style="border-top: 1px solid black;"> <td style="border-right: 1px solid black; padding: 5px 10px;"><math>a</math></td> <td style="padding: 5px 10px;"><math>a</math></td> <td style="padding: 5px 10px;"><math>a</math></td> </tr> <tr> <td style="border-right: 1px solid black; padding: 5px 10px;"><math>b</math></td> <td style="padding: 5px 10px;"><math>a</math></td> <td style="padding: 5px 10px;"><math>a</math></td> </tr> </table>	$\cdot$	$a$	$b$	$a$	$a$	$a$	$b$	$a$	$a$
$+$	$a$	$b$																		
$a$	$b$	$\star$																		
$b$	$\star$	$\star$																		
$\cdot$	$a$	$b$																		
$a$	$a$	$a$																		
$b$	$a$	$a$																		

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Find a two element **counterexample** to the following argument:

$\frac{fg(x) \approx gf(x)}{f(x) \approx g(x)}$	<table style="border-collapse: collapse; margin: auto;"> <tr> <td style="border-right: 1px solid black; padding: 5px 10px;"></td> <td style="padding: 5px 10px;"><math>f</math></td> </tr> <tr style="border-top: 1px solid black;"> <td style="border-right: 1px solid black; padding: 5px 10px;"><math>a</math></td> <td style="padding: 5px 10px;"><math>a</math></td> </tr> <tr> <td style="border-right: 1px solid black; padding: 5px 10px;"><math>b</math></td> <td style="padding: 5px 10px;"><math>b</math></td> </tr> </table>		$f$	$a$	$a$	$b$	$b$	<table style="border-collapse: collapse; margin: auto;"> <tr> <td style="border-right: 1px solid black; padding: 5px 10px;"></td> <td style="padding: 5px 10px;"><math>g</math></td> </tr> <tr style="border-top: 1px solid black;"> <td style="border-right: 1px solid black; padding: 5px 10px;"><math>a</math></td> <td style="padding: 5px 10px;"><math>b</math></td> </tr> <tr> <td style="border-right: 1px solid black; padding: 5px 10px;"><math>b</math></td> <td style="padding: 5px 10px;"><math>a</math></td> </tr> </table>		$g$	$a$	$b$	$b$	$a$
	$f$													
$a$	$a$													
$b$	$b$													
	$g$													
$a$	$b$													
$b$	$a$													

Give a derivation using Birkhoff's Rules, stating reasons for your steps, to show that

$$x \cdot y \approx u \cdot v \vdash (x \cdot x) + (y \cdot y) \approx (y \cdot y) + (x \cdot x).$$

The following is a derivation:

1. $x \cdot y \approx u \cdot v$	Given
2. $x \cdot x \approx y \cdot y$	1 Subs
3. $(x \cdot x) + (y \cdot y) \approx (y \cdot y) + (y \cdot y)$	2 Repl
4. $y \cdot y \approx x \cdot x$	2 Symm
5. $(y \cdot y) + (y \cdot y) \approx (y \cdot y) + (x \cdot x)$	4 Repl
6. $(x \cdot x) + (y \cdot y) \approx (y \cdot y) + (x \cdot x)$	3,5 Trans

Fill in the reasons for the steps in the following derivation. [This is part of a derivation found by an automated theorem prover.]

1.	$x + 0$	$\approx$	$x$	given
2.	$x + (-x)$	$\approx$	$0$	given
3.	$x + y$	$\approx$	$y + x$	given
4.	$(x + y) + z$	$\approx$	$x + (y + z)$	given
5.	$x \cdot (y + z)$	$\approx$	$(x \cdot y) + (x \cdot z)$	given
6.	$(x + y) \cdot z$	$\approx$	$(x \cdot z) + (y \cdot z)$	given
7.	$x \cdot x$	$\approx$	$x$	given
8.	$(x \cdot y) + (x \cdot z)$	$\approx$	$x \cdot (y + z)$	5 Symm
9.	$(x \cdot z) + (y \cdot z)$	$\approx$	$(x + y) \cdot z$	6 Symm
10.	$(-x) + x$	$\approx$	$x + (-x)$	3 Subs
11.	$(-x) + x$	$\approx$	$0$	10, 2 Trans
12.	$(x + (-x)) + y$	$\approx$	$x + ((-x) + y)$	4 Subs
13.	$x + ((-x) + y)$	$\approx$	$(x + (-x)) + y$	12 Symm
14.	$(x + (-x)) + y$	$\approx$	$0 + y$	2 Repl
15.	$0 + y$	$\approx$	$y + 0$	3 Subs
16.	$y + 0$	$\approx$	$y$	1 Subs
17.	$0 + y$	$\approx$	$y$	15, 16 Trans
18.	$(x + (-x)) + y$	$\approx$	$y$	14, 17 Trans
19.	$x + ((-x) + y)$	$\approx$	$y$	13, 18 Trans
20.	$((-x) + x) + y$	$\approx$	$(x + (-x)) + y$	10 Repl
21.	$((-x) + x) + y$	$\approx$	$y$	20, 18 Trans
22.	$((-x) + x) + y$	$\approx$	$(-x) + (x + y)$	4 Subs
23.	$(-x) + (x + y)$	$\approx$	$((-x) + x) + y$	22 Symm
24.	$(-x) + (x + y)$	$\approx$	$y$	23, 21 Trans
25.	$y + x$	$\approx$	$x + y$	3 Subs (or Symm)
26.	$(-x) + (y + x)$	$\approx$	$(-x) + (x + y)$	25 Repl
27.	$(-x) + (y + x)$	$\approx$	$y$	26, 24 Trans
28.	$(x \cdot x) + (x \cdot y)$	$\approx$	$x \cdot (x + y)$	8 Subs
29.	$(x \cdot x) + (x \cdot y)$	$\approx$	$x + (x \cdot y)$	7 Repl