Equivalent Formulas

In the following table you are asked to consider whether or not certain formulas are equivalent. The $\Box$s in the left column are to be replaced by the appropriate binary connective in each of the successive columns. For example, consider the row starting with $P \Box P \sim P$. Now go over to the column with the header $\wedge$. The question is whether or not $P \wedge P \sim P$. If the formulas are equivalent, put a check mark ($\surd$) in the box. Otherwise leave the box empty.

	$\vee$	$\wedge$	$\rightarrow$	$\leftrightarrow$	$\mid$	$\curlywedge$
$P \Box P \sim P$	$\surd$	$\surd$				
$P \Box (P \Box P) \sim (P \Box P) \Box P$	$\surd$	$\surd$		$\surd$	$\surd$	$\surd$
$P \Box Q \sim Q \Box P$	$\surd$	$\surd$		$\surd$	$\surd$	$\surd$
$P \Box (Q \Box P) \sim P$						
$P \Box (Q \Box R) \sim (P \Box Q) \Box R)$	$\surd$	$\surd$		$\surd$		

Adequate Sets of Connectives

Circle (or highlight) the formulas among $0, 1, P, \neg P$ that can be represented by a formula $F(P)$ using (only) the connectives in $\mathcal{C}$:

Given				
$\mathcal{C} = \{\vee\}$	0	1	$\boxed{P}$	$\neg P$
$\mathcal{C} = \{\rightarrow\}$	0	$\boxed{1}$	$\boxed{P}$	$\neg P$
$\mathcal{C} = \{\leftrightarrow, \rightarrow\}$	0	$\boxed{1}$	$\boxed{P}$	$\neg P$
$\mathcal{C} = \{\wedge, 1\}$	0	$\boxed{1}$	$\boxed{P}$	$\neg P$

Circle (or highlight) the connectives that can be realized using the connectives in $\mathcal{C}$:

Given				
$\mathcal{C} = \{\rightarrow, \vee\}$	$\boxed{\vee}$	$\wedge$	$\boxed{\rightarrow}$	$\leftrightarrow$
$\mathcal{C} = \{\wedge, \leftrightarrow\}$	$\vee$	$\boxed{\wedge}$	$\rightarrow$	$\boxed{\leftrightarrow}$
$\mathcal{C} = \{\neg, \wedge\}$	$\boxed{\vee}$	$\boxed{\wedge}$	$\boxed{\rightarrow}$	$\boxed{\leftrightarrow}$
$\mathcal{C} = \{\rightarrow, \leftrightarrow\}$	$\vee$	$\wedge$	$\boxed{\rightarrow}$	$\boxed{\leftrightarrow}$

Substitution/Replacement

In each of the following inferences you are to choose the most inclusive answer for how the inference could be accomplished. The four choices are: **substitution**, **replacement**, **both**, **neither**.

1. $$\frac{\neg\neg P \ \sim\ P}{\neg\neg\neg P \ \sim\ \neg P}$$ *Both*

2. $$\frac{P \ \sim\ Q}{P \to P \ \sim\ Q \to Q}$$ *Substitution*

3. $$\frac{P \wedge \neg Q \ \sim\ \neg(P \to Q)}{Q \wedge \neg P \ \sim\ \neg(Q \to P)}$$ *Substitution*

4. $$\frac{P \vee Q \ \sim\ Q \vee P}{(P \vee Q) \to (P \vee Q) \ \sim\ (Q \vee P) \to (P \vee Q)}$$ *Replacement*

5. $$\frac{P \to (Q \to P) \ \sim\ 1}{(Q \to P) \to (P \to (Q \to P)) \ \sim\ 1}$$ *Substitution*

More on Adequate Connectives

Determine if the binary connective $\triangle$ defined by $P \triangle Q \ \sim\ P \wedge \neg Q$ is adequate. Give Reasons! [Either show that some adequate set of connectives can be expressed using $\triangle$, or prove that some propositional formula cannot be expressed using $\triangle$.]

Consider the formulas $\mathsf{F}(P)$ that one can express using $\triangle$.

One certainly has P; and also 0 since $P \triangle P \sim 0$.

However we cannot obtain $\neg P$ since applying $\triangle$ to P and 0 gives a formula that is equivalent to one of these two:

$$\begin{aligned} P\triangle P &\sim 0 \\ P\triangle 0 &\sim P \\ 0\triangle P &\sim 0 \\ 0\triangle 0 &\sim 0 \end{aligned}$$

Suppose you know that $\mathsf{F}(P, Q, R, S)$ and $\mathsf{G}(P, Q, R, S)$ are two propositional formulas such that $\mathsf{F} \wedge \mathsf{G}$ has the same truth table as $\mathsf{F} \vee \mathsf{G}$. Prove that F and G have the same truth tables.
PROOF:
(First Proof): Using Fundamental Equivalences one has

$$\begin{array}{rcl} \mathsf{F} & \sim & \mathsf{F} \vee (\mathsf{F} \wedge \mathsf{G}) \\ & \sim & \mathsf{F} \vee (\mathsf{F} \vee \mathsf{G}) \\ & \sim & (\mathsf{F} \vee \mathsf{F}) \vee \mathsf{G} \\ & \sim & \mathsf{F} \vee \mathsf{G} \end{array} \qquad \begin{array}{rcl} \mathsf{G} & \sim & \mathsf{G} \vee (\mathsf{G} \wedge \mathsf{F}) \\ & \sim & \mathsf{G} \vee (\mathsf{F} \wedge \mathsf{G}) \\ & \sim & \mathsf{G} \vee (\mathsf{F} \vee \mathsf{G}) \\ & \sim & \mathsf{G} \vee (\mathsf{G} \vee \mathsf{F}) \\ & \sim & (\mathsf{G} \vee \mathsf{G}) \vee \mathsf{F} \\ & \sim & \mathsf{G} \vee \mathsf{F} \\ & \sim & \mathsf{F} \vee \mathsf{G} \end{array}$$

Thus one has $\mathsf{F} \sim \mathsf{G}$, so they have the same truth tables.

(Second Proof): This proof examines the truth tables. The value of $\mathsf{F} \vee \mathsf{G}$ at any row $\vec{e}$ of the truth table is $\max(\mathsf{F}(\vec{e}), \mathsf{G}(\vec{e}))$; and the value of $\mathsf{F} \wedge \mathsf{G}$ at any row $\vec{e}$ of the truth table is $\min(\mathsf{F}(\vec{e}), \mathsf{G}(\vec{e}))$. By assumption $\mathsf{F} \vee \mathsf{G}$ has the same truth table as $\mathsf{F} \wedge \mathsf{G}$, so, for every row $\vec{e}$,

$$\max(\mathsf{F}(\vec{e}), \mathsf{G}(\vec{e})) \ = \ \min(\mathsf{F}(\vec{e}), \mathsf{G}(\vec{e})).$$

This can only happen if $\mathsf{F}(\vec{e}) = \mathsf{G}(\vec{e})$ holds for each row $\vec{e}$, that is, the truth tables are the same.

Give an **inductive definition** of $\mathit{Num}(\mathsf{F})$, the number of occurrences of variables in a propositional formula F.

(Answer):

$$\begin{array}{rcl} \mathit{Num}(P) & = & 1 \\ \mathit{Num}(0) & = & 0 \\ \mathit{Num}(1) & = & 0 \\ \\ \mathit{Num}(\neg\,\mathsf{F}) & = & \mathit{Num}(\mathsf{F}) \\ \mathit{Num}(\mathsf{F} \vee \mathsf{G}) & = & \mathit{Num}(\mathsf{F}) + \mathit{Num}(\mathsf{G}) \\ \mathit{Num}(\mathsf{F} \wedge \mathsf{G}) & = & \mathit{Num}(\mathsf{F}) + \mathit{Num}(\mathsf{G}) \\ \mathit{Num}(\mathsf{F} \rightarrow \mathsf{G}) & = & \mathit{Num}(\mathsf{F}) + \mathit{Num}(\mathsf{G}) \\ \mathit{Num}(\mathsf{F} \leftrightarrow \mathsf{G}) & = & \mathit{Num}(\mathsf{F}) + \mathit{Num}(\mathsf{G}) \end{array}$$