The Expansion Theorem

Prove that if a term $F(X)$ in the language of classes is always equal to $AX \cup BX'$, where A and B are classes, then $A = F(1)$ and $B = F(0)$.
PROOF:

Letting $X = 1$ gives

$$F(1) \;=\; A1 \cup B1' \;=\; A \cup 0 \;=\; A,$$

and then letting $X = 0$ gives

$$F(0) \;=\; A0 \cup B0' \;=\; 0 \cup B \;=\; B.$$

For the formula $F(A, B) = A \cup A'B$ carry out the following computations to calculate the expansion of $F(A, B)$ on A, B:

	Full Expression		Value
$F(1,1) =$	$1 \cup 1'1$	$=$	1
$F(1,0) =$	$1 \cup 1'0$	$=$	1
$F(0,1) =$	$0 \cup 0'1$	$=$	1
$F(0,0) =$	$0 \cup 0'0$	$=$	0

Thus expanding on A, B gives: $F(A, B) \;=\;$ $AB \cup AB' \cup A'B$

For the formula $F(A, B, C) = (A \cup B)'(B \cup C)$ carry out the following computations to calculate the expansion of $F(A, B, C)$ on B:

	Full Expression		Simplified
$F(A,1,C) =$	$(A \cup 1)'(1 \cup C)$	$=$	0
$F(A,0,C) =$	$(A \cup 0)'(0 \cup C)$	$=$	$A'C$

Expanding on B gives: $F(A, B, C) \;=\;$ $A'CB'$

For the formula $F(A,B,C) = (A \cup B)(BC)'$ carry out the following computations to calculate the expansion of $F(A,B,C)$ on A, C:

	Full Expression		Simplified
$F(1,B,1) =$	$(1 \cup B)(B1)'$	$=$	B'
$F(1,B,0) =$	$(1 \cup B)(B0)'$	$=$	1
$F(0,B,1) =$	$(0 \cup B)(B1)'$	$=$	0
$F(0,B,0) =$	$(0 \cup B)(B0)'$	$=$	B

Thus expanding on A, C gives: $F(A,B,C) \;=\; \underline{\quad B'AC \cup AC' \cup BA'C' \quad}$

Elimination

Given two classes A and B, prove that there is a class X such that $AX \cup BX' = 0$ holds if and only if $AB = 0$, that is, iff the intersection of A and B is empty.
PROOF:

First let us suppose that $AX \cup BX' = 0$, for some X. Then one has

$$\begin{aligned} AX &= 0 \\ BX' &= 0. \end{aligned}$$

We can rewrite these equations as

$$\begin{aligned} A &\subseteq X' \\ B &\subseteq X. \end{aligned}$$

But then

$$A \cap B \;\subseteq\; X' \cap X \;=\; 0$$

so

$$A \cap B \;=\; 0.$$

This proves one direction.

Now let us suppose that $AB = 0$ and show that there is an X such that $AX \cup BX' = 0$. This means we want $AX = 0$ and $BX' = 0$, which can be expressed by saying that we want $AX = 0$ and $B \subseteq X$. Clearly if we select $X = B$ then these conditions are satisfied.

For the formula $E(A,B,C) = (A \cup B \cup C)(A' \cup B' \cup C')$ carry out the following computations to eliminate B from the equation $E(A,B,C) = 0$:

	Full Expression		Simplified
$E(A,1,C) =$	$(A \cup 1 \cup C)(A' \cup 1' \cup C')$	$=$	$A' \cup C'$
$E(A,0,C) =$	$(A \cup 0 \cup C)(A' \cup 0' \cup C')$	$=$	$A \cup C$

Eliminating B gives (simplify first!): $AC' \cup A'C = 0$

For the formula $E(A,B,C,D) = (A \cup B)(C \cup D)$ carry out the following computations to eliminate A, D from the equation $E(A,B,C,D) = 0$:

	Full Expression		Simplified
$E(1,B,C,1) =$	$(1 \cup B)(C \cup 1)$	$=$	1
$E(1,B,C,0) =$	$(1 \cup B)(C \cup 0)$	$=$	C
$E(0,B,C,1) =$	$(0 \cup B)(C \cup 1)$	$=$	B
$E(0,B,C,0) =$	$(0 \cup B)(C \cup 0)$	$=$	BC

Eliminating A, D gives (simplify first!): $BC = 0$

Fill in the following tree to give a proof of the validity of the argument using the method of Lewis Carroll. Be sure to give the number of the reason for each boxed letter.

1 $CDE = 0$

2 $AB'C' = 0$

3 $D'EF' = 0$

4 $A'GE = 0$

5 $B'D'F = 0$

$B'GE = 0$