Put the following formula in prenex form. Use capital letters when renaming variables. Show details of your work:

$$\forall u\,(u < y) \to \exists y \forall z\,\left[\neg\,(u + y < w + z) \to \neg\,\exists w\,(w \cdot y < w \cdot z)\right]$$

Answer: $\exists U\,\exists Y\,\forall z\,\forall W\,\Big((U < y) \to \left[\neg\,(u + Y < w + z) \to \neg\,(W \cdot Y < W \cdot z)\right]\Big)$

DETAILS: First rename bound occurrences of u, y, z:

$$\forall U\,(U < y) \to \exists Y\,\forall z\,\left[\neg\,(u + Y < w + z) \to \neg\,\exists W\,(W \cdot Y < W \cdot z)\right]$$

Then proceed to pull the quantifiers to the front:

$$\exists U\,\Big((U < y) \to \exists Y\,\forall z\,\left[\neg\,(u + Y < w + z) \to \neg\,\exists W\,(W \cdot Y < W \cdot z)\right]\Big)$$

$$\exists U\,\exists Y\,\forall z\,\Big((U < y) \to \left[\neg\,(u + Y < w + z) \to \neg\,\exists W\,(W \cdot Y < W \cdot z)\right]\Big)$$

$$\exists U\,\exists Y\,\forall z\,\Big((U < y) \to \left[\neg\,(u + Y < w + z) \to \forall W\,\neg\,(W \cdot Y < W \cdot z)\right]\Big)$$

$$\exists U\,\exists Y\,\forall z\,\forall W\,\Big((U < y) \to \left[\neg\,(u + Y < w + z) \to \neg\,(W \cdot Y < W \cdot z)\right]\Big)$$

Convert the following argument into a set $\mathcal{C}$ of clauses such that the argument is valid iff the set $\mathcal{C}$ is not satisfiable. Use capital letters when changing the names of variables. Show details of your work.

$$\forall x\Big(\exists y\,(x \cdot y < x) \;\rightarrow \exists z\,\big((y \cdot z < x) \wedge \neg\,(y \cdot x < z)\big)\Big) \quad \therefore\; \exists x \forall y\,(x \cdot y < y)$$

Answer: $\{\neg\,(x \cdot Y < x),\; (y \cdot fxY < x)\}$
$\{\neg\,(x \cdot Y < x),\; \neg\,(y \cdot x < fxY)\}$
$\{\neg\,(x \cdot gx < gx)\}$
gives the clauses of $\mathcal{C}$.

DETAILS: There is one premiss and the conclusion. Changing bound variables in the premiss gives:

$$\forall x\Big(\exists Y\,(x \cdot Y < x) \;\rightarrow \exists z\,\big((y \cdot z < x) \wedge \neg\,(y \cdot x < z)\big)\Big)$$

Then the prenex form is:

$$\forall x\,\forall Y\,\exists z\,\Big((x \cdot Y < x) \;\rightarrow \big((y \cdot z < x) \wedge \neg\,(y \cdot x < z)\big)\Big)$$

Skolemizing the prenex form of the premiss gives

$$\forall x\,\forall Y\,\Big((x \cdot Y < x) \;\rightarrow \big((y \cdot fxY < x) \wedge \neg\,(y \cdot x < fxY)\big)\Big)$$

Putting the matrix in conjunctive form gives

$$\big((\neg\,(x \cdot Y < x) \;\vee (y \cdot fxY < x)\big) \wedge \big(\neg\,(x \cdot Y < x) \;\vee \neg\,(y \cdot x < fxY)\big) \tag{1}$$

The negated conclusion is

$$\neg\,\exists x \forall y\,(x \cdot y < y)$$

which is equivalent to

$$\forall x \exists y\,\neg\,(x \cdot y < y)$$

Skolemizing the prenex form of the negated conclusion gives

$$\forall x\,\neg\,(x \cdot gx < gx)$$

The matrix in conjunctive form is given by

$$\neg\,(x \cdot gx < gx) \tag{2}$$

From (1) and (2) we have the three clauses given in the Answer above.