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The magnetic pendulum and weather

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Abstract

The magnetic pendulum represents an intriguing demonstration of chaotic motion and unpredictability. Because of this it provides an excellent tool to illustrate the challenges associated with weather prediction. Presented in this note are numerical simulations of various paths traced out by the magnetic pendulum which serve to highlight the well-known 'butterfly effect'. A mathematical model and computer program to solve the equations is provided to allow for further exploration. An analogy between this demonstration and weather prediction is also given.

Keywords: chaos, magnetic pendulum, weather, numerical simulations

Supplementary material for this article is available online

1. Introduction

The magnetic pendulum, shown in figure 1 is a popular desktop toy having a small magnet at the end of the pendulum and additional magnets on the base of the apparatus. When released from rest it swings freely under the influence of gravity and the various magnets positioned on the base, and will eventually come to rest over one of the magnets. This demonstration illustrates an important signature of non-linear systems known as 'sensitivity to initial conditions' or the 'butterfly effect', a term coined by Edward Lorenz [1].

The demonstration will illustrate the complicated path followed by the pendulum before coming to rest over one of the magnets. In fact, it is easily observed that if the pendulum is repeatedly released from initial positions that are close to one another it will likely settle over a different magnet each time. This scenario is captured in the next section using numerical simulations to track the trajectory of the pendulum.

2. Mathematical model and numerical simulations

The motion of the pendulum was modelled using the following set of second-order differential equations [2]

$$\frac{d^2x}{dt^2}+R\frac{dx}{dt}+Cx=\sum_{i=1}^{3}\frac{(x_i-x)}{[(x_i-x)^2+(y_i-y)^2+D^2]^{3/2}}\,, \tag{1}$$

$$\frac{d^2y}{dt^2}+R\frac{dy}{dt}+Cy=\sum_{i=1}^{3}\frac{(y_i-y)}{[(x_i-x)^2+(y_i-y)^2+D^2]^{3/2}}\,. \tag{2}$$

Here, $(x(t), y(t))$ denotes the position of the pendulum at time t in the $x-y$ plane which coincides with the base of the apparatus. The curve traced out by all such points $(x(t), y(t))$ for $0 \leq t < \infty$ corresponds to the trajectory that would be observed when viewed from above. The parameter R is a measure of the damping force while

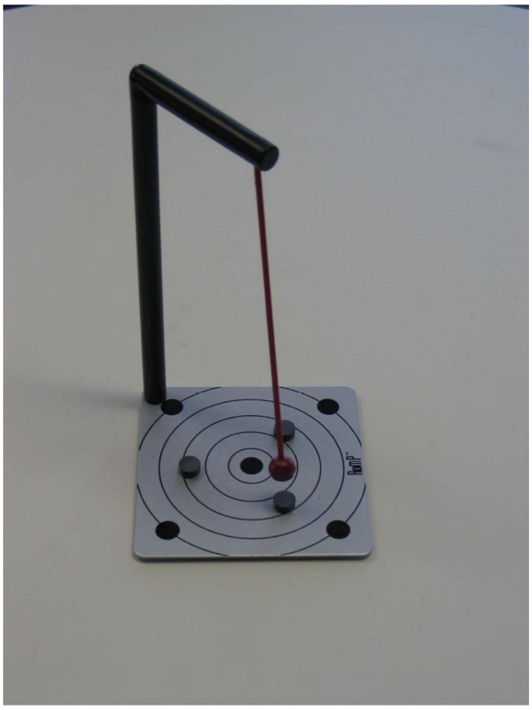

Figure 1. The magnetic pendulum.

the parameter C can be interpretted as a spring constant associated with the gravitational restoring force. The right-hand sides of equations (1) and (2) represent the magnetic force of attraction which is inversely proportional to the distance squared from each magnet. The positions of the magnets are denoted by (x_i, y_i) where $i = 1, 2, 3$ and the parameter D refers to the minimum distance between the pendulum and the base of the apparatus.

The system of equations (1) and (2) was solved numerically using the fourth-order Runge–Kutta (RK4) algorithm [3]. Although there are several Runge–Kutta algorithms, the RK4 method was used because of its simplicity, accuracy and popularity. To implement the RK4 algorithm equations (1) and (2) must first be expressed as a system of first-order differential equations. This can be achieved by introducing $u = dx/dt$ and $v = dy/dt$. Then the system given by equations (1) and (2) now becomes

$$\frac{dx}{dt} = u \ , \tag{3}$$

$$\frac{du}{dt} = -Ru - Cx + \sum_{i=1}^{3} \frac{(x_i - x)}{[(x_i - x)^2 + (y_i - y)^2 + D^2]^{3/2}} \ , \tag{4}$$

$$\frac{dy}{dt} = v \ , \tag{5}$$

$$\frac{dv}{dt} = -Rv - Cy + \sum_{i=1}^{3} \frac{(y_i - y)}{[(x_i - x)^2 + (y_i - y)^2 + D^2]^{3/2}} \ . \tag{6}$$

To understand how the RK4 scheme works, consider the following generic single first-order differential equation

$$\frac{dz}{dt} = f(z, t) \text{ subject to } z(0) = z_0 \ ,$$

for some function $f(z, t)$. The RK4 scheme advances the solution from time t_n to time $t_{n+1} = t_n + \Delta t$ using the marching algorithm given by

$$z_{n+1} = z_n + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4) \ , \tag{7}$$

where

$$k_1 = f(z_n, t_n) \ , \ k_2 = f\left(z_n + \frac{k_1 \Delta t}{2}, t_n + \frac{\Delta t}{2}\right) \ ,$$

$$k_3 = f\left(z_n + \frac{k_2 \Delta t}{2}, t_n + \frac{\Delta t}{2}\right) \ ,$$

$$k_4 = f(z_n + k_3 \Delta t, t_n + \Delta t) \ .$$

Thus, starting with the initial value z_0, subsequent values $z_1, z_2, \cdots$ can easily be generated by using the recursion relation (7). Here, z_n is the computed solution at time t_n while z_{n+1} is the sought after solution at time t_{n+1}. This algorithm can easily be extended to the system of differential equations given by (3)–(6). The MATLAB program used to solve equations (3)–(6) is provided as supplementary material[1] (stacks.iop.org/PED/55/063002/mmedia).

Using the MATLAB program provided along with the values $R = .2$, $C = .5$, $D = .25$ and

[1] See MATLAB code submitted as supplemental material which was used to numerically solve the equations of motion.

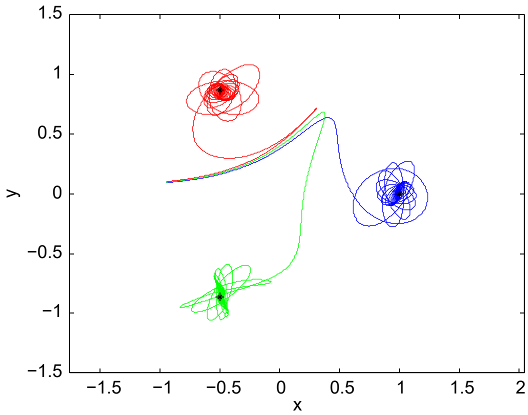

Figure 2. Three computed pendulum trajectories shown in different colours. The black asterisks denote the base magnet locations. The initial positions of the three pendulum trajectories were taken to be (-.95,.09), (-.95,.095), (-.95,.1).

$\Delta t = .01$ simulations were conducted for the case where the pendulum was released from rest, that is, $u(0) = v(0) = 0$. Plotted in figure 2 are three trajectories shown in the $x - y$ plane. Each trajectory is indicated by a different colour and the initial positions of these trajectories, as indicated in the figure caption, were taken to be close to one another. The diagram reveals that a small change in the initial position causes the pendulum to come to rest over a different magnet. In other words, in order to predict where the pendulum will end up one would need to know the initial position very accurately, and if this location is off by just a little then the prediction would be wrong. This is an example of what is commonly referred to as the 'butterfly effect', that is, a butterfly flapping its wings in one part of the world can change the weather in another part of the world. It is because of this simple concept that makes it difficult to predict the weather far into the future. In order to predict the weather far into the future one would need to know the initial state of the atmosphere very accurately. For example, at each point in the atmosphere one would need to know the air speed, temperature and pressure and if there is an error in any one of these quantities at any one location then that error would be responsible for limiting how far into the future the prediction would be meaningful. Because small changes in the initial state of the atmosphere can lead to dramatically different outcomes, the weather becomes unpredictable over time.

The above illustration represents an example of chaos. Many other possibilities can be explored using the program. For example, a map can be constructed whereby each point in the $x - y$ plane can be coloured in such a way that it is indicative of where the pendulum comes to rest when released from rest from that point. Other possibilities could include using non-zero initial speeds or different parameter values for R, C and D. Coding in MATLAB can even be explored. For example, the program can be modified so that equations (3)–(6) are solved using the second-order Runge–Kutta (RK2) algorithm instead of the fourth-order Runge–Kutta (RK4) algorithm. A lot can be learned through these numerical experiments!

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Serge D'Alessio completed his undergraduate studies in Engineering Physics at McMaster University and received his Ph.D. in Applied Mathematics from Western University. He is a professor in the Centre for Education in Mathematics and Computing housed in the Faculty of Mathematics at the University of Waterloo, and actively visits high schools across Canada and abroad to promote and motivate the application of mathematics in the sciences and engineering. He is also a licensed professional engineer with research interests in fluid mechanics. In his spare time he enjoys cycling, swimming, running and hiking.