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Torricelli's law revisited

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Abstract

Suppose a tank initially full of water is drained through a small hole at the bottom. At what rate does the water level drop? How long does it take to empty the tank? A mathematical model is formulated to answer these questions. An experiment is also conducted to check if the prediction agrees with observation. A novel modification to the mathematical model is presented to bring theory and observation in harmony. This involves accounting for the non-uniform velocity profile through the exit hole. Unsteady effects resulting from the acceleration of the fluid is also explored theoretically. The problem and solution procedure is suitable for an introductory physics course.

Keywords: Torricelli's law, Bernoulli's principle, fluids

S Supplementary material for this article is available online

(Some figures may appear in colour only in the online journal)

1. Introduction

Estimating the time to drain a tank, pond or reservoir is an important problem in engineering and the applied sciences. Because of this it has received numerous investigations. We begin by summarizing some previous studies. Various experimental studies ([1–9]) have been conducted and several approaches have been advanced to bring theory and experiment in agreement. For example, one approach involves introducing a coefficient of discharge, C_D, defined as the ratio of the actual discharge to the theoretical discharge which can be determined experimentally ([3, 7]). Another method accounts for the area of the exit hole when compared to the cross-sectional area of the tank ([2, 5, 8]). If the area of the exit hole is not significantly smaller than that of the tank, then the fluid acceleration can no longer be ignored. Including an energy loss term is another strategy ([6]). For very small exit holes the effect of wetting can also be responsible

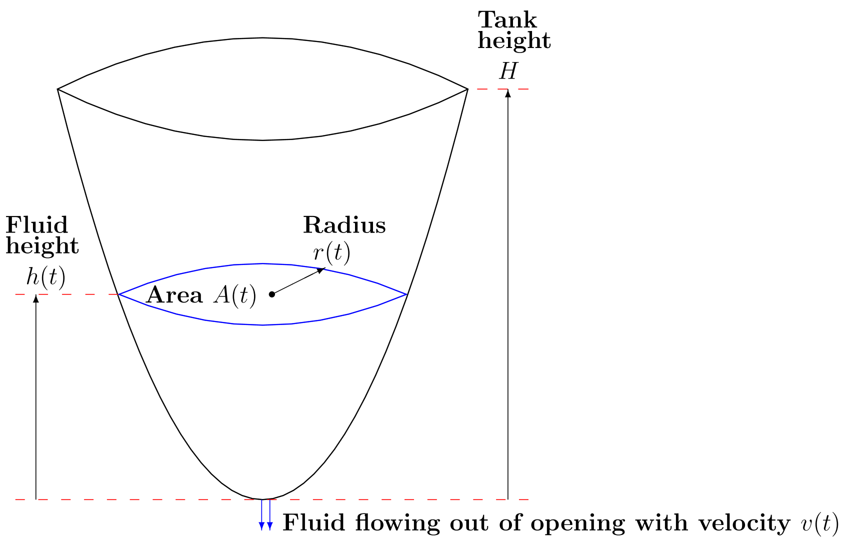

Figure 1. Schematic illustration of the problem.

for disagreements between theory and measurements ([9]). Wetting refers to the ability of a fluid to remain in contact with a solid surface, and is the result of the intermolecular forces acting between the fluid and the surface. The time to empty a tank can also be determined computationally, as in [10], by numerically solving the Navier–Stokes equations which are the governing equations for fluid motion. Although this approach can capture the fine details in the flow pattern, this level of accuracy is not always needed or required.

The present investigation attempts to tackle the drainage problem both theoretically and experimentally using techniques that are accessible to an undergraduate student. A mathematical model is first derived and solved for different tank configurations. Then, comparisons are made with a simple experiment for a specific tank configuration. Agreement between theory and experiment is made by proposing a coefficient of discharge. Here, we offer a new physical interpretation of the coefficient of discharge in terms of the fluid velocity profile through the exit hole. In addition, we also present a full exact solution for unsteady flow which to our knowledge is missing in the literature.

2. Mathematical formulation

Consider a tank having the shape of a cylindrical bowl with height H as shown in figure 1. Water is allowed to escape with velocity v through a small hole having an area a at the bottom. At time t the water has a surface area A and water level denoted by h. Since we are dealing with a cylindrical tank the cross-sectional area will be circular, and thus $A = \pi r^2$ where r is the radius at time t. In a small time Δt the water level will drop by Δh. Because the density is constant the principle of conservation of mass is equivalent to conservation of volume, and so the change in water volume in the tank must equal that leaving the tank. In differential form this can be written as follows

$$-A\Delta h = av\Delta t,$$

where the minus sign indicates that h is decreasing. As $\Delta t \to 0$ the above can be expressed as a differential equation (DE) given by

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{av}{A}. \tag{1}$$

Although the area of the hole, a, is constant, v, A and of course h will vary with time t. In order to solve (1) we need to specify how v and A depend on h or t.

We can use *Torricelli's law* [11] to relate v to h. Evangelista Torricelli, who lived from 1608 to 1647, was an Italian physicist and mathematician. In addition to observing principles of fluid flow, he is also remembered for his invention of the barometer and for his work on estimating the value of the acceleration due to gravity. This law states that if we ignore all forms of friction the fluid velocity as it exits the hole will be given by $v = \sqrt{2gh}$ which corresponds to the velocity a drop of fluid would have if released from rest from a height h. Actually, Torricelli's law follows from *Bernoulli's equation* [11] which for steady flow can be expressed as either

$$P + \rho gh + \frac{1}{2}\rho v^2 = \text{constant} \quad \text{or} \quad P_1 + \rho gh_1 + \frac{1}{2}\rho v_1^2 = P_2 + \rho gh_2 + \frac{1}{2}\rho v_2^2. \tag{2}$$

Here, P is the pressure, ρ is the fluid density, and g is the acceleration due to gravity. The subscripts 1 and 2 refer to different points along the path of a fluid parcel. In the absence of friction, this equation is essentially a statement of conservation of energy for an ideal, or inviscid, fluid. If we take point 1 to be the water surface and point 2 to coincide with the hole, then $P_1 = P_2 = P_{\text{atm}}$ where P_{atm} refers to atmospheric pressure, and $h_1 - h_2 = h$. Setting $v_1 = 0$ (to be explained shortly) yields $v_2 = \sqrt{2gh}$ which is Torricelli's Law.

Let us first consider the special case where A is constant, that is, the tank is a right-circular cylinder. If we make the approximation that the water level falls slowly in comparison with the emergent velocity as it exits the hole, then $v_1 \approx 0$, and from Torricelli's Law equation (1) becomes

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{a\sqrt{2gh}}{A},$$

which is a nonlinear separable first-order DE that one finds in many textbooks, such as [12, 13]. Separating the variables we obtain

$$\int_H^0 \frac{\mathrm{d}h}{\sqrt{h}} = -\frac{a\sqrt{2g}}{A}\int_0^T \mathrm{d}t.$$

Integrating (noting that the integral on the left-hand-side is an improper integral) yields the following expression for the time T to drain the entire tank

$$T = \frac{A}{a}\sqrt{\frac{2H}{g}}. \tag{3}$$

The height of the water level as a function of time can also easily be determined and is given by

$$h(t) = \left(\sqrt{H} - \frac{a\sqrt{2g}}{2A}t\right)^2.$$

We can even account for the moving water level by applying Bernoulli's equation. In this case instead of making the approximation $v_1 \approx 0$ we set $v_1 = \mathrm{d}h/\mathrm{d}t$. Using equation (2) this leads to

$$v_2 = \sqrt{2gh + \left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2}. \tag{4}$$

Inserting (4) into equation (1) for v and solving yields the time

$$T = \sqrt{\frac{2H}{g}\left(\frac{A^2}{a^2} - 1\right)}. \tag{5}$$

In the limit that $A^2/a^2 \gg 1$, then (5) reduces to (3) as expected. Thus, the impact of accounting for the moving water level is to reduce the time. This makes sense since the velocity given by (4) exceeds that given by Torricelli's law. It is important to emphasize the underlying steady flow assumption. In reality, this is an approximation since the flow will be unsteady. The steady flow assumption/approximation refers to a situation when the tank drains slowly meaning that the acceleration of the falling liquid surface is much smaller than the acceleration due to gravity. As explained in [5, 8], if $A^2/a^2 \gg 1$ this is a good approximation. In the appendix we present the full solution corresponding to unsteady flow.

Another common tank configuration is the conical or funnel-shaped tank. We take the top radius to be R_1 and the bottom radius to be $R_0 \ll R_1$. For this configuration the height, h, is related to the radius, r, through the relation

$$h = \frac{H(r - R_0)}{(R_1 - R_0)},$$

where H is the height of the tank. Using

$$A = \pi r^2 = \pi\left(R_0 + \frac{(R_1 - R_0)h}{H}\right)^2, \quad a = \pi R_0^2, \quad v = \sqrt{2gh}.$$

Equation (1) becomes

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{\sqrt{2gh}}{\left(1 + \frac{(R_1 - R_0)h}{R_0 H}\right)^2}.$$

Again, separating the variables and integrating leads to the following expression for the time T to empty the full tank

$$T = \sqrt{\frac{2H}{g}}\left(\frac{8}{15} + \frac{4R_1}{15R_0} + \frac{3}{15}\left[\frac{R_1}{R_0}\right]^2\right).$$

We note that if $R_1 = R_0$ then $a = A$ and the fluid will accelerate out of the bottom with acceleration g, and thus, the flow is clearly unsteady. Here, the mathematical model breaks down since the assumption $A^2/a^2 \gg 1$ is violated. This case is addressed in the appendix.

3. The clepsydra

We next consider the general case where the shape of the tank is given by $h = cr^n$ where c and n are constants. Then it follows that

$$A = \frac{\pi}{c^{\frac{2}{n}}} h^{\frac{2}{n}}.$$

Using Torricelli's law equation (1) again leads to a nonlinear separable first-order DE given by

$$\frac{dh}{dt} = -\frac{ac^{\frac{2}{n}}\sqrt{2g}}{\pi} h^{\frac{n-4}{2n}}.$$

Although this equation is straight forward to solve, we will focus on a particular shape. We notice that if $n = 4$ the above simplifies to

$$\frac{dh}{dt} = -\frac{a\sqrt{2gc}}{\pi} = \text{constant}.$$

Hence, the water level falls at a constant rate, and so this tank shape corresponds to a water clock, or *clepsydra*. Water clocks represent an ancient time-measuring device invented in Egypt. The time to empty the tank is easily shown to be

$$T = \frac{\pi H}{a\sqrt{2gc}}. \tag{6}$$

It is worth noting that if A in equation (3) is set equal to the area at the top of the tank (i.e. when $h = H$), then the time given by (6) is exactly half that of (3). In other words, it takes twice as long to drain a right-circular cylindrical tank than it does to drain a clepsydra that fits exactly inside the right-circular cylindrical tank.

An alternate approach to discovering the interesting property of the clepsydra can be achieved by combining volumes of revolution and related rates as illustrated in the following two-part exercise.

Exercise. A bowl of height h is obtained by rotating the curve $y = x^4$ about the y-axis.

(a) Show that the volume of water, $V(h)$, that the bowl can hold is given by

$$V(h) = \frac{2\pi}{3} h^{\frac{3}{2}}.$$

(b) Suppose the bowl is filled with water and allowed to leak from a small hole in the bottom. If the rate at which the volume of water leaking out of the hole per unit time is given by $\alpha\sqrt{h}$ where α is a constant, show that the rate dh/dt at which the water level drops in the bowl is constant.

4. Experimental validation and discussion

An experiment was designed to see how closely the observed time to drain a tank agrees with the predicted time given by (6). A video of the experiment, which is available for viewing, was created and is provided as supplemental material. The experimental apparatus, shown in figure 2, consisted of a plastic cylindrical tank, a wooden stand, and a beaker to capture the drained water. A clepsydra-shaped bowl given by $h = cr^4$ was accurately machined from

Figure 2. Experimental set up.

a cylindrical section of plastic. The values of various parameters including the tank dimensions are: $H = 16$ cm, $a = \pi/16$ cm^2, $c = 1/16$ cm^{-3}, $g = 981$ cm s^{-2}. For this configuration $A^2/a^2 = 256 \gg 1$ where A refers to the area at the top of the tank. Thus, the steady flow assumption is a reasonable approximation. Of course, as the water level nears the bottom of the tank the acceleration of the fluid becomes more important. Using these values the predicted, or theoretical, time is $T = 23.1$ s. By placing a ruler beside the tank and measuring how long it took the water to fall by 1 cm at various instances during the draining we were able to confirm that the water level fell at a nearly uniform rate. However, after running numerous experiments the average observed time to empty the tank was $T = 29.2$ s which exceeds the predicted value by 26.4%, and is well beyond any experimental error. This suggests that the observed rate of fall deviates considerably from the theoretical rate. It is worth mentioning that including the effect of the moving water level cannot resolve the discrepancy because as we already saw for the right-circular cylinder it will decrease the predicted time, and hence will worsen the agreement.

In an attempt to improve the agreement between theory and observation a modification to the mathematical model was made. This involved accounting for a non-uniform velocity profile through the opening at the bottom as shown in figure 3. It is well known that the fluid velocity along a surface or wall is zero. This is a result of friction brought on by viscous forces and is referred to as the *no-slip condition* [14]. Thus, we propose the following empirical power-law velocity profile through the opening given by

$$\frac{v(r)}{v_0} = \left[1 - \left(\frac{r}{R}\right)^k\right] \quad \text{where } v_0 = \sqrt{2gh}, \tag{7}$$

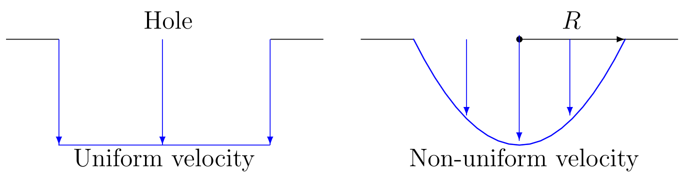

Figure 3. Velocity profiles through the opening.

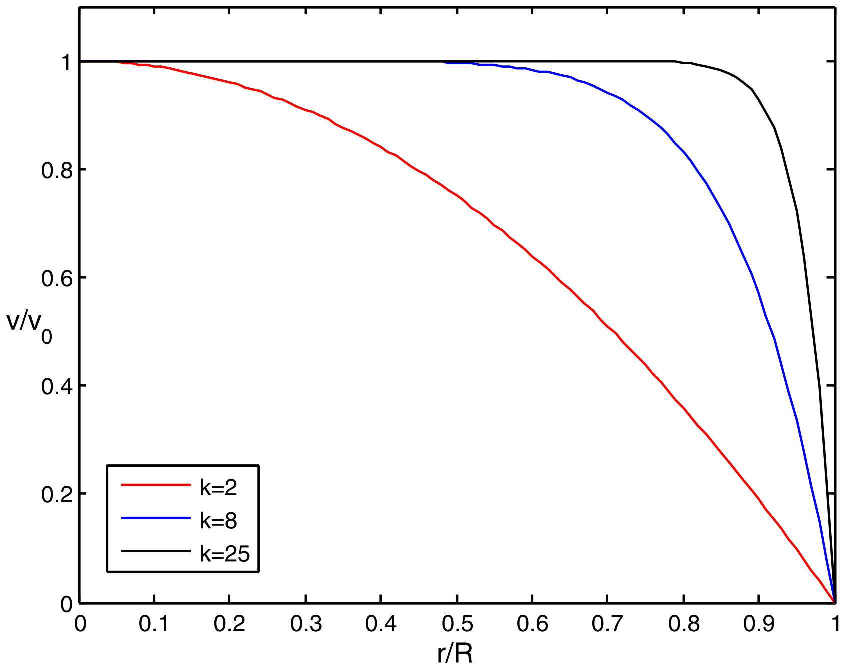

Figure 4. Empirical velocity profiles.

where R denotes the radius of the hole and k is a free parameter. This empirical formula incorporates a lot of physics. For example, we notice that $v = 0$ when $r = R$ in order to comply with the no-slip condition. Also, for laminar, or streamlined, flow it yields the exact parabolic profile which corresponds to $k = 2$ [14]. Lastly, for turbulent, or chaotic, flow we expect the (time-averaged) profile to become flatter near the center and steeper at the wall which are characteristic features of turbulent flow. As shown in figure 4, equation (7) is able to mimic such a profile as k increases. In fact, in the limit as $k \rightarrow \infty$ we obtain the uniform profile $v = \sqrt{2gh}$ given by Torricelli's law.

Up to this point the effects of friction have been ignored. Inside the tank and away from the opening at the bottom this is justified because of the significantly smaller fluid velocity and larger radius. However, at the exit the fluid velocity will be much greater due to the smaller opening, and so the effects of friction are no longer negligible. Since frictional losses are proportional to the gradient in velocity, the losses from the walls inside the tank are expected to be much smaller than the losses at the exit. Using the power-law velocity profile given by (7), the average velocity through the opening, denoted by $\bar{v}$, can be determined as follows

$$\bar{v} = \frac{1}{\pi R^2}\int_0^R 2\pi r v(r)\mathrm{d}r = \frac{2\sqrt{2gh}}{R^2}\int_0^R r\left[1-\left(\frac{r}{R}\right)^k\right]\mathrm{d}r = \left(\frac{k}{k+2}\right)\sqrt{2gh}. \tag{8}$$

So, instead of using the velocity $v = \sqrt{2gh}$ in equation (1), it makes more sense to use $v = \bar{v}$. In order for the theoretical time to be in agreement with the observed time we require that

$$\frac{k}{k+2} = \frac{\text{theoretical time}}{\text{observed time}} \approx 0.8.$$

Solving this yields $k = 8$. We emphasize that the value of k will depend on the tank configuration. Once the k value for a given tank is known, it can be used to predict important quantities such as the rate at which the water level falls, the height of the water level at a given time, and more. The k value is also indicative of the nature of the flow through the exit hole. For example, if k is close to 2 then the flow is laminar; however, if the value is large then the flow is highly turbulent. Since $k = 8$ in our case, we are dealing with turbulent flow. The effect of the assumed velocity profile is to reduce the exit velocity through the opening by a factor of $C_D = k/(k+2)$. It is conceivable that for some tank configurations the coefficient of discharge can be close to unity, but in general $C_D < 1$. A worthwhile laboratory experiment would be to determine k for various tank configurations. For example, this could include using tanks in the shape of a right-circular cylinder having the same tank diameter and height, but different exit hole diameters, or funnel-shaped tanks which are readily available in hardware stores.

5. Summary

This investigation revisited the drainage problem both experimentally and theoretically. Although the experimental set up was very simple, it was easy to perform in a laboratory and did not require sophisticated equipment. Also, most experimental studies have focused on a cylindrical tank whereas here we presented a clepsydra-shaped tank. The new contributions arising from this study are twofold. First, we provided an alternate interpretation of the coefficient of discharge by relating it to the non-uniform velocity profile through the exit hole. An empirical profile was proposed which can reproduce the exact profile for laminar flow and can mimic turbulent profiles as well. Second, the influence of the fluid acceleration was explored theoretically. An exact solution to the general unsteady problem was found and it was shown that as the ratio of the area of the exit hole to that of the tank shrinks to zero the unsteady effects vanish. Lastly, the approach and techniques used in this study are fully accessible to an undergraduate student in physics.

Acknowledgments

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Appendix

Here, we consider the unsteady flow of an inviscid fluid through a small hole of area a at the bottom of a cylindrical tank having a height H and a constant cross-sectional area A. It can be shown ([2, 5, 8, 14]) that the instantaneous height of the fluid above the bottom, $h(t)$, satisfies the following nonlinear second-order DE

$$h\left(g+\frac{\mathrm{d}^2h}{\mathrm{d}t^2}\right)=\frac{1}{2}\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2\left[\frac{A^2}{a^2}-1\right]. \tag{A1}$$

As noted in [5], if $g \gg \mathrm{d}^2h/\mathrm{d}t^2$ then the left-hand-side of (A1) can be replaced by simply gh and the solution of (A1) yields a drainage time T given by equation (5). This amounts to ignoring the acceleration of the fluid, and thus, is termed quasi-steady flow. We now consider the case when the term $\mathrm{d}^2h/\mathrm{d}t^2$ is not negligible.

As done in [2], we first cast equation (A1) in dimensionless form by introducing

$$\eta=\frac{h}{H},\quad \tau=\sqrt{\frac{g}{2\beta H}}t\quad \text{where } \beta=\frac{A^2}{a^2}-1,$$

with the understanding that $\beta > 1$ and usually $\beta \gg 1$. Then equation (A1) becomes

$$2\eta\frac{\mathrm{d}^2\eta}{\mathrm{d}\tau^2}-\beta\left(\frac{\mathrm{d}\eta}{\mathrm{d}\tau}\right)^2+4\beta\eta=0. \tag{A2}$$

We want to solve equation (A2) subject to the initial conditions

$$\eta=1,\quad \frac{\mathrm{d}\eta}{\mathrm{d}\tau}=0 \text{ at } \tau=0.$$

Setting $V=\mathrm{d}\eta/\mathrm{d}\tau$ to denote the dimensionless velocity and using the Chain rule to write

$$\frac{\mathrm{d}V}{\mathrm{d}\tau}=\frac{\mathrm{d}V}{\mathrm{d}\eta}\frac{\mathrm{d}\eta}{\mathrm{d}\tau}=V\frac{\mathrm{d}V}{\mathrm{d}\eta}=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(V^2\right),$$

transforms equation (A2) to the linear first-order DE given by

$$\frac{\mathrm{d}\chi}{\mathrm{d}\eta}-\frac{\beta}{\eta}\chi=-4\beta, \tag{A3}$$

where $\chi=V^2$. The solution to (A3) satisfying $\chi(1)=0$ (which follows from the initial condition $V(\eta=1)=0$) is easily found to be

$$\chi(\eta)=\frac{4\beta}{(\beta-1)}\left[\eta-\eta^\beta\right].$$

Hence,

$$V=\frac{\mathrm{d}\eta}{\mathrm{d}\tau}=\pm\mu\sqrt{\eta-\eta^\beta}\quad \text{where }\ \mu=\sqrt{\frac{4\beta}{\beta-1}}. \tag{A4}$$

Since $\mathrm{d}\eta/\mathrm{d}\tau<0$ we will take the negative root. This also means that $\eta<1$ for all $\tau>0$. We note that in the limit that $\beta\to\infty$ the dimensionless velocity approaches $V=-2\sqrt{\eta}$ (since $\eta^\beta\to 0$) which is the dimensionless version of equation (1) after applying Torricelli's law. From this we observe that the term η^β in equation (A4) accounts for the acceleration of the fluid. We also note that the solution given by (A4) is in full agreement with that obtained in [8].

We next solve equation (A4) to determine η which extends the work presented in [8]. We begin by separating the variables

$$\int_{\eta_0}^{\eta} \frac{\mathrm{d}x}{\sqrt{x - x^{\beta}}} = -\mu \int_0^{\tau} \mathrm{d}t,$$

and recall from the Binomial series that

$$\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{\infty} \frac{(2n)!x^n}{2^{2n}(n!)^2} \quad \text{for } |x| < 1.$$

Thus,

$$\frac{1}{\sqrt{x - x^{\beta}}} = \frac{1}{\sqrt{x}} \frac{1}{\sqrt{1 - x^{\beta-1}}} = \frac{1}{\sqrt{x}} \sum_{n=0}^{\infty} \frac{(2n)!x^{n(\beta-1)}}{2^{2n}(n!)^2} = \sum_{n=0}^{\infty} \frac{(2n)!x^{n(\beta-1)-\frac{1}{2}}}{2^{2n}(n!)^2},$$

for $|x| < 1$. Inserting this series into the above integral and integrating term-by-term leads to

$$\sum_{n=0}^{\infty} \frac{(2n)!}{\left(n(\beta-1)+\frac{1}{2}\right) 2^{2n}(n!)^2} \left[\eta^{n(\beta-1)+\frac{1}{2}} - \eta_0^{n(\beta-1)+\frac{1}{2}}\right] = -\mu\tau.$$

We note that the above series converges provided $\eta < 1$. For this reason we replaced the initial condition $\eta(0) = 1$ with $\eta(0) = \eta_0$ where $\eta_0 < 1$ and can be made arbitrarily close to unity. This guarantees that $\eta \leqslant \eta_0 < 1$ for all $\tau \geqslant 0$. Expanding the first two terms of the series we obtain

$$2[\sqrt{\eta} - \sqrt{\eta_0}] + \frac{1}{(2\beta-1)} \left[\eta^{\beta-\frac{1}{2}} - \eta_0^{\beta-\frac{1}{2}}\right] + \cdots = -\sqrt{\frac{4\beta}{\beta-1}}\tau,$$

where the first term on the left-hand-side represents the solution for quasi-steady flow while all the remaining terms are contributions arising from the acceleration of the fluid. As a check, if we let $\beta \to \infty$ then $\mu \to 2$ and only the first term in the series survives; the expansion then reduces to

$$2[\sqrt{\eta} - \sqrt{\eta_0}] = -2\tau.$$

Setting $\eta_0 = 1$ yields $\eta = (1-\tau)^2$ which corresponds to dimensionless version of the solution

$$h(t) = \left(\sqrt{H} - \frac{a\sqrt{2g}}{2A}t\right)^2,$$

presented earlier, which is the expected result.

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