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A mathematical model of the velocity of a cyclist riding over uneven terrain

Serge D'Alessio[*]

Centre for Education in Mathematics and Computing, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, N2L 3G1, Canada

E-mail: sdalessio@uwaterloo.ca

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Abstract

Derived in this article is a mathematical model of a cyclist riding over uneven terrain. The model is very general; it is expressed in terms of arbitrary functions that describe how the elevation changes with horizontal distance and how the cyclist's power output varies, and also accounts for road friction and air resistance. The model was validated by considering special cases such as horizontal and inclined surfaces. Although the governing equation is too complicated to solve exactly for the general case, a numerical solution procedure is proposed and tested. In addition, an approximate analytical solution procedure is outlined. As an illustration some numerical results are presented for the case of a cyclist riding over rolling hills. The adopted approach is well suited for undergraduate students in mathematical physics. The formulation equally applies to a vehicle driving over a variable landscape. This work also unites concepts from physics with mathematical modelling and coding. Lastly, a video abstract is also provided.

Keywords: cycling, mathematical model, air resistance, road friction, variable landscape, numerical simulations, analytical solution

S Supplementary material for this article is available online

(Some figures may appear in colour only in the online journal)

1. Introduction

Mathematical modelling is an effective tool that can be used to simulate various problems and outcomes. In this investigation a novel two-dimensional mathematical model is proposed to simulate a cyclist riding over a varying terrain. The model accounts for road friction and air

[*]Author to whom any correspondence should be addressed.

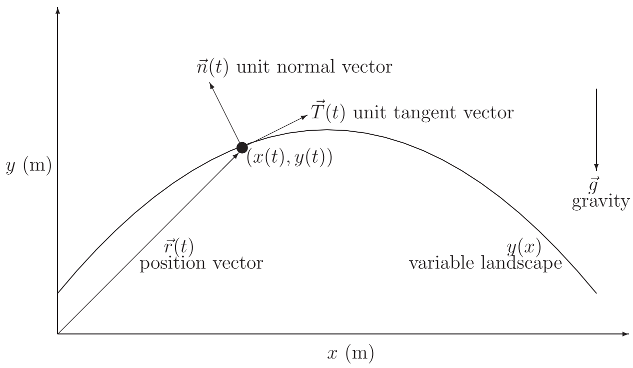

Figure 1. The position vector $\vec{r}(t)$ of a cyclist riding along a variable landscape $y(x)$.

resistance and is expressed in terms of a user-specified terrain. The cyclist's power output can also be specified. Such a model can assist cyclists in their training over various landscapes. Cyclists can count on having two dependable training partners: the force of the wind and the force of gravity. Both of these are captured in this model.

The application of science in sport has evolved significantly over the years. In particular, the sport of cycling has benefitted tremendously from the works of Burke [1], Burke and Newsom [2], Gregor and Conconi [3], and Glaskin [4], to list a few. More recently, Dahmen, Wolf and Saupe [5] have formulated a mathematical optimization problem by blending a mechanical model of cycling with a simple physiological model for the exertion of an athlete to compute optimal riding strategies for time trials on mountain ascents. The fundamentals and scientific principles covered in these studies have guided this research and helped shape the proposed mathematical model which will be presented shortly.

The paper is structured as follows. In the next section the problem is formulated mathematically and expressed as a nonlinear differential equation for the cyclist's position and velocity with time. In section 3 various special cases are considered which admit exact solutions. An analytical technique for constructing an approximate solution is also presented to tackle the general case. Then in section 4 a numerical solution procedure is outlined and used to simulate a cyclist riding over a sinusoidally varying terrain. Section 5 is devoted to summarizing the study. Although the model was derived to simulate the velocity of a cyclist, it can also be used to simulate that of a vehicle driving over a variable landscape. Further, this investigation also emphasizes the importance of scientific computation and analytical techniques.

2. Mathematical formulation

We consider a cyclist riding along an uneven road. At each instant in time the position of the cyclist is specified using a Cartesian coordinate system. As illustrated in figure 1, for two-dimensional motion the position vector of the cyclist is given by

$$\vec{r}(t) = (x(t), y(t)), \tag{1}$$

where $x(t)$ and $y(t)$ are the horizontal and vertical positions, respectively, of the cyclist at time t.

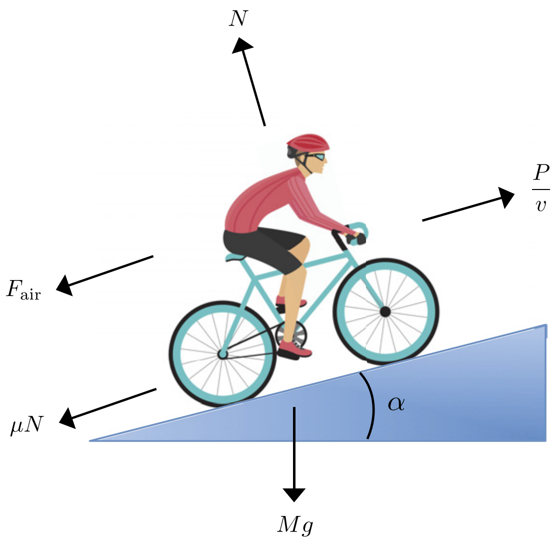

Figure 2. A sketch illustrating the various forces acting on the cyclist.

The velocity, $\dot{\vec{r}}$, and acceleration, $\ddot{\vec{r}}$, vectors then become

$$\dot{\vec{r}} = (\dot{x}, y'\dot{x}), \tag{2}$$

$$\ddot{\vec{r}} = (\ddot{x}, y'\ddot{x} + y''\dot{x}^2), \tag{3}$$

where dots represent derivatives with respect to t while the primes refer to derivatives with respect to x. The unit tangent, $\vec{T}$, and normal, $\vec{n}$, vectors are

$$\vec{T} = \frac{(1, y')}{\sqrt{1 + (y')^2}} \qquad \vec{n} = \frac{(-y', 1)}{\sqrt{1 + (y')^2}}, \tag{4}$$

with $\vec{T}$ pointing in the direction of motion and $\vec{n}$ pointing into the air. Applying Newton's second law of motion yields

$$\vec{F} = M\ddot{\vec{r}}, \tag{5}$$

where $\vec{F}$ denotes the net force and M is the combined mass of the cyclist and bicycle. The various forces acting on the cyclist as shown in figure 2 include

$$\vec{F} = \vec{F}_{\text{air}} + \frac{P}{v}\vec{T} + M\vec{g} + N\vec{n} - \mu N\vec{T},$$

where $\vec{F}_{\text{air}}$ is the aerodynamic force, P is the cyclist's power output, $v = |\dot{\vec{r}}| = \dot{x}\sqrt{1 + (y')^2}$ is the speed, $M\vec{g} = (0, -Mg)$ is the gravitational force with g denoting the acceleration due to gravity, N is the normal force, and μ is the coefficient of static rolling friction which will depend on the road type and tire. Here, we have ignored drive train friction and have absorbed the effect of dynamic rolling resistance into the aerodynamic force. It is assumed that all the power generated by the cyclist is transmitted to the wheels, that is, the efficiency is 100%.

At low velocities air resistance is well approximated by the relation $\vec{F}_{\text{air}} = -kv\vec{T}$ where k is the coefficient of air resistance and its value depends on the dimensions and shape of the cyclist and the density of air (Meade [6]). Often, values for the quantity $\frac{k}{M}$ are reported. If wind is present then air resistance will be proportional to the difference between the velocity of the cyclist and the wind velocity, denoted by $\vec{V}$, and is given by $\vec{F}_{\text{air}} = -k(v - V_{\text{w}})\vec{T}$. Here, V_{w} is the component of the wind velocity in the direction of motion of the cyclist; it is positive for a tailwind and negative for a headwind. Although a cross wind can have a swaying effect on a cyclist, theoretically it does no work and therefore will be ignored. As expected, a tailwind reduces air resistance, and when the tailwind exceeds the cyclist's velocity then air resistance will actually assist the cyclist rather than oppose the motion. At higher velocities air resistance is proportional to the square of the velocity and is given by $\vec{F}_{\text{air}} = -\frac{1}{2}\rho A C_{\text{d}} v^2 \vec{T}$ where ρ is the density of air, A is the frontal area of the cyclist and bicycle, and C_{d} is the drag coefficient (Timmerman and van der Weele [7]). The wind velocity can easily be incorporated by replacing v^2 with $(v - V_{\text{w}})^2$ and if $v \gg V_{\text{w}}$ we can simply ignore the effect of the wind. Changes in air density and gravity as a result of cycling up or down a hill are negligible and will be ignored. The actual formula to be used for air resistance will depend on the specific situation we are considering and can be written as $\vec{F}_{\text{air}} = -f_{\text{air}}\vec{T}$ where $f_{\text{air}} = k(v - V_{\text{w}})$ or $f_{\text{air}} = \frac{1}{2}\rho A C_{\text{d}}(v - V_{\text{w}})^2$.

We next decompose equation (5) in the directions of $\vec{T}$ and $\vec{n}$. Since the vectors $\vec{T}$ and $\vec{n}$ are orthogonal, it immediately follows that

$$\ddot{\vec{r}} = (\ddot{\vec{r}} \cdot \vec{T})\vec{T} + (\ddot{\vec{r}} \cdot \vec{n})\vec{n} = \frac{[(1 + (y')^2)\ddot{x} + y'y''\dot{x}^2]}{\sqrt{1 + (y')^2}}\vec{T} + \frac{y''\dot{x}^2}{\sqrt{1 + (y')^2}}\vec{n},$$

$$\vec{g} = (\vec{g} \cdot \vec{T})\vec{T} + (\vec{g} \cdot \vec{n})\vec{n} = -\frac{gy'}{\sqrt{1 + (y')^2}}\vec{T} - \frac{g}{\sqrt{1 + (y')^2}}\vec{n}.$$

By examining the forces in the $\vec{n}$ direction we obtain

$$N = \frac{M(g + y''\dot{x}^2)}{\sqrt{1 + (y')^2}}. \tag{6}$$

Substituting this result for N into the equation in the $\vec{T}$ direction leads to the following second-order, nonlinear differential equation

$$[1 + (y')^2]\ddot{x} + y'y''\dot{x}^2 = -\frac{f_{\text{air}}}{M}\sqrt{1 + (y')^2} + \frac{P}{M\dot{x}} - gy' - \mu(g + y''\dot{x}^2). \tag{7}$$

For a given terrain $y(x)$ and a specified power output P (which could vary with time, t) equation (7) can be solved for $x(t)$ subject to initial conditions for x and $\dot{x}$. Listed in table 1 are parameter values that will be used in this study.

3. Analytical solutions

In this section we present some special cases which admit exact solutions and an approximate analytical solution procedure for the general case.

Table 1. Values used for various parameters.

Parameter	Symbol	Value
Acceleration due to gravity	g	$9.8\ \mathrm{m\,s^{-2}}$
Combined mass	M	85 kg
Air density	ρ	$1.2\ \mathrm{kg\,m^{-3}}$
Frontal area	A	$0.3\ \mathrm{m^2}$
Drag coefficient	C_d	0.8
Coefficient of air resistance	k/M	$0.025\ \mathrm{s^{-1}}$
Coefficient of rolling friction	μ	0.025

3.1. Level terrain

In this situation $y = y' = y'' = 0$ and equation (7) reduces to

$$M\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{P}{v} - f_{\mathrm{air}} - \mu Mg, \tag{8}$$

where $v = \dot{x}$. An exact solution to (8) has been obtained and will be presented in section 3.4. The solution is complicated and not very insightful. The important feature of the solution is that if a cyclist starts from rest and rides into a headwind with constant power P, then with $f_{\mathrm{air}} = k(v - V_{\mathrm{w}})$ the cyclist quickly reaches a steady-state velocity, v_{s}, given by

$$v_{\mathrm{s}} = -\frac{1}{2}\left(\frac{\mu gM}{k} - V_{\mathrm{w}}\right) + \sqrt{\frac{P}{k} + \frac{1}{4}\left(\frac{\mu gM}{k} - V_{\mathrm{w}}\right)^2}.$$

Using the values in table 1 along with $P = 250$ W and $V_{\mathrm{w}} = -3\ \mathrm{m\,s^{-1}}$ ($10.8\ \mathrm{km\,hr^{-1}}$) (i.e. a headwind) we find that $v_{\mathrm{s}} \approx 6.2\ \mathrm{m\,s^{-1}}$ ($22.3\ \mathrm{km\,hr^{-1}}$). With a tailwind of $V_{\mathrm{w}} = 3\ \mathrm{m\,s^{-1}}$ under the same conditions $v_{\mathrm{s}} \approx 8.0\ \mathrm{m\,s^{-1}}$ ($28.7\ \mathrm{km\,hr^{-1}}$).

3.2. Uphill climbing velocity

Here, $y = x\tan\alpha, y' = \tan\alpha, y'' = 0$ where $\tan\alpha$ denotes the constant uphill slope. Setting $u = \dot{x}\ \sec\ \alpha$, the uphill velocity, then equation (7) becomes

$$M\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{P}{u} - f_{\mathrm{air}} - Mg\ \sin\ \alpha - \mu Mg\ \cos\ \alpha. \tag{9}$$

If we ignore air resistance (i.e. $f_{\mathrm{air}} = 0$) and suppose a cyclist can generate a constant power output of P while climbing, then the cyclist will eventually attain a constant climbing velocity, u_{c}, given by

$$u_{\mathrm{c}} = \frac{1}{(\sin\ \alpha + \mu\ \cos\ \alpha)}\left(\frac{P}{Mg}\right).$$

We see that the climbing velocity is proportional to the ratio $\frac{P}{Mg}$ which is known as the *power-to-weight ratio*. It immediately follows that a lighter cyclist has a clear advantage. Using the values in table 1 along with $\alpha = 7°$ (which corresponds to a grade of 12.3% where the %grade $= 100\tan\alpha$) and $P = 250$ W we find that $u_{\mathrm{c}} \approx 2.0\ \mathrm{m\,s^{-1}}$ ($7.4\ \mathrm{km\,hr^{-1}}$).

3.3. Downhill coasting velocity

Suppose a cyclist starts from rest on top of a hill having a constant inclination of α and height H, and coasts down the hill without pedalling (i.e. $P = 0$) or applying the breaks. In this case $y = -x\tan\alpha, y' = -\tan\alpha, y'' = 0$; again setting $u = \dot{x}\sec\alpha$, the downhill velocity, then with $f_{\text{air}} = \frac{1}{2}\rho A C_{\text{d}} u^2$ (i.e. zero or negligible wind) equation (7) simplifies to

$$M\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{1}{2}\rho A C_{\text{d}} u^2 + Mg\sin\alpha - \mu Mg\cos\alpha. \tag{10}$$

Using the Chain Rule we can write $\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\mathrm{d}u}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = u\frac{\mathrm{d}u}{\mathrm{d}s}$ where s refers to the distance travelled down the incline. Equation (10) can then be cast in the form

$$u\frac{\mathrm{d}u}{\mathrm{d}s} = \gamma - \Gamma u^2 \quad \text{where} \quad \gamma = g(\sin\alpha - \mu\cos\alpha) \quad \text{and} \quad \Gamma = \frac{\rho A C_{\text{d}}}{2M}.$$

Separating the variables leads to

$$\int_0^{u_{\text{f}}} \frac{u\,\mathrm{d}u}{\gamma - \Gamma u^2} = \int_0^{\frac{H}{\sin\alpha}} \mathrm{d}s,$$

and integrating yields the final velocity, u_{f}, given by

$$u_{\text{f}} = \sqrt{\frac{\gamma}{\Gamma}\left(1 - \exp\left(-\frac{2H\Gamma}{\sin\alpha}\right)\right)}.$$

The terminal velocity, denoted by u_{T}, occurs when the left-hand side of equation (10) is zero. This means that the forces on the right-hand side exactly balance and the cyclist travels at a constant velocity given by

$$u_{\text{T}} = \sqrt{\frac{\gamma}{\Gamma}}.$$

As expected $\lim_{H\to\infty} u_{\text{f}} = u_{\text{T}}$. Using the values in table 1 along with $\alpha = 7^\circ$ and $H = 100$ m we find that $u_{\text{f}} \approx 22.9$ m s^{-1} (82.6 km hr^{-1}) while $u_{\text{T}} \approx 23.7$ m s^{-1} (85.3 km hr^{-1}). Recall that the terminal velocity of a skydiver in a belly-to-earth free fall position is about 54 m s^{-1} (195 km hr^{-1}).

Repeating this calculation with $f_{\text{air}} = ku$ leads to the following equation for u_{f}

$$\left(\frac{k}{M}\right)u_{\text{f}} + \gamma\ln\left[1 - \left(\frac{k}{M}\right)\frac{u_{\text{f}}}{\gamma}\right] + \left(\frac{k}{M}\right)^2\frac{H}{\sin\alpha} = 0,$$

which can only be solved numerically. Again, using the values from table 1 together with $\alpha = 7^\circ$, $H = 100$ m and Newton's method (Burden and Faires [8]) to estimate the root yields $u_{\text{f}} \approx 27.2$ m s^{-1} (97.9 km hr^{-1}) which exceeds the value found above since the air resistance has been reduced. Here, the terminal velocity is given by

$$u_{\text{T}} = \left(\frac{M}{k}\right)\gamma,$$

and computes to $u_{\text{T}} \approx 38.0$ m s^{-1} (137 km hr^{-1}).

As a final case we will ignore air resistance and use the principle of conservation of energy. We will also account for the rotational kinetic energy of the wheels. Conservation of energy requires that

$$MgH = \frac{1}{2}Mu_{\mathrm{f}}^2 + 2\left(\frac{1}{2}I\omega^2\right) + \frac{\mu MgH}{\tan\,\alpha},$$

where the term on the left-hand side represents the potential energy at the top of the hill and the terms on the right-hand side represent the translational kinetic energy, rotational kinetic energy and energy lost to road friction, respectively, at the bottom of the hill. Here, I is the moment of inertia of a wheel and ω is the angular velocity of the wheel. If we approximate the wheel as a thin disk, then $I = \frac{1}{2}m_{\mathrm{w}}R^2$ where R is the radius of the wheel and m_{w} is its mass. Also, $\omega = u_{\mathrm{f}}/R$ and thus $I\omega^2 = \frac{1}{2}m_{\mathrm{w}}u_{\mathrm{f}}^2$. Solving the above equation for u_{f} gives

$$u_{\mathrm{f}} = \sqrt{2gH\left(1 - \frac{\mu}{\tan\,\alpha}\right)\left(\frac{M}{M + m_{\mathrm{w}}}\right)}.$$

Since $M \gg m_{\mathrm{w}}$ we can safely neglect the rotational kinetic energy of the wheels and obtain

$$u_{\mathrm{f}} \approx \sqrt{2gH\left(1 - \frac{\mu}{\tan\,\alpha}\right)}.$$

Using the same values as above we find that $u_{\mathrm{f}} \approx 39.5$ m s^{-1} (142 km hr^{-1}). It comes as no surprize that the final velocity surpasses the values found in the previous cases.

3.4. Slowly varying terrain

Since an exact solution to equation (7) is out of reach we propose a strategy to obtain an approximate analytical solution. For cycling considerations it is fair to say that a chosen riding route can be classified as slowly varying. By this we mean that the vertical length scale is significantly smaller than the horizontal length scale, or equivalently the slope at any location is small. To put this in perspective a gradient of 10% is considered to be fairly steep for a cyclist, while for the expansion we are about to use this is sufficiently small. We can exploit this slowly varying property by casting equation (7) in dimensionless form. Let L denote a horizontal length scale, h a vertical length scale, and U a velocity scale. Then

$$x = Lx^*, \qquad y = hy^*, \qquad t = \frac{L}{U}t^*, \qquad P = \frac{MU^3}{L}P^*, \qquad f_{\mathrm{air}} = \frac{MU^2}{L}f_{\mathrm{air}}^*,$$

where the asterisk denotes a dimensionless quantity. In dimensionless form and dropping the asterisks for notational convenience equation (7) becomes

$$[1 + \varepsilon^2(y')^2]\ddot{x} + \varepsilon^2 y'y''\dot{x}^2 = -f_{\mathrm{air}}\sqrt{1 + \varepsilon^2(y')^2} + \frac{P}{\dot{x}} - \varepsilon\beta y' - \mu(\beta + \varepsilon y''\dot{x}^2), \tag{11}$$

where $\varepsilon = \frac{h}{L} \ll 1$ and $\beta = \frac{gL}{U^2}$. Since ε is a small parameter we can construct an approximate solution for x using a regular perturbation method (Logan [9]). This involves seeking a solution for x in the form of a series given by

$$x = x_0 + \varepsilon x_1 + \cdots.$$

Substituting this into (11) produces a hierarchy of problems at various orders of ε. The function $y(x) = y(x_0 + \varepsilon x_1 + \cdots)$ can be expanded using a Taylor series as follows

$$y(x) = y(x_0 + \varepsilon x_1 + \cdots) = y(x_0) + \varepsilon x_1 y'(x_0) + \cdots .$$

Lastly, the initial conditions, $x(0)$ and $\dot{x}(0)$, need to be expressed as

$$x(0) = x_0(0) + \varepsilon x_1(0) + \cdots , \qquad \dot{x}(0) = \dot{x}_0(0) + \varepsilon \dot{x}_1(0) + \cdots .$$

Since $x(0)$ and $\dot{x}(0)$ are constants independent of ε, it follows that $x_0(0) = x(0)$, $\dot{x}_0(0) = \dot{x}(0)$ and $x_1(0) = \dot{x}_1(0) = \cdots = 0$.

The leading-order problem satisfies

$$\dot{v}_0 = -f_{\text{air}} + \frac{P}{v_0} - \mu\beta, \tag{12}$$

and represents the dimensionless version of equation (8) with $v_0 = \dot{x}_0$. This corresponds to the solution over level terrain. Taking $f_{\text{air}} = \Gamma_1(v - V_{\text{w}})$ where $\Gamma_1 = \frac{kL}{MU}$ equation (12) becomes

$$\dot{v}_0 = -\frac{\Gamma_1}{v_0}\left[\left(v_0 - \frac{c}{2\Gamma_1}\right)^2 - K\right], \tag{13}$$

where

$$c = \Gamma_1 V_{\text{w}} - \mu\beta, \qquad K = \frac{P}{\Gamma_1} + \frac{c^2}{4\Gamma_1^2}.$$

For the case when P is constant the exact solution to (13) is given by

$$\frac{\left(v_0 - \frac{c}{2\Gamma_1}\right)^2 - K}{\left(v_i - \frac{c}{2\Gamma_1}\right)^2 - K} = \left[\frac{\left(v_i - \frac{c}{2\Gamma_1} + \sqrt{K}\right)\left(v_0 - \frac{c}{2\Gamma_1} - \sqrt{K}\right)}{\left(v_i - \frac{c}{2\Gamma_1} - \sqrt{K}\right)\left(v_0 - \frac{c}{2\Gamma_1} + \sqrt{K}\right)}\right]^{\frac{c}{2\Gamma_1\sqrt{K}}} \mathrm{e}^{-2\Gamma_1 t},$$

where $v_i = v_0(0)$ is the initial velocity. As discussed in section 3.1, this transient solution will quickly approach the steady-state velocity given by

$$v_{\text{s}} = \frac{c}{2\Gamma_1} + \sqrt{K}.$$

The solution for $x_0(t)$ can be obtained by integrating $v_0(t)$ and imposing the initial condition $x_0(0) = 0$. This will yield a complicated expression for $x_0(t)$.

The first-order problem is governed by the equation

$$\dot{v}_1 + \left(\Gamma_1 + \frac{P}{v_0^2}\right) v_1 = -\beta y'(x_0) - \mu y''(x_0) v_0^2, \tag{14}$$

and satisfies the initial conditions $x_1 = v_1 = 0$ at $t = 0$ where $v_1 = \dot{x}_1$. The terms involving $y'(x_0)$ and $y''(x_0)$ on the right-hand side account for an uneven terrain. Based on the solution for v_0, an exact solution to (14) for v_1 will be very complicated. We can, however, make analytical progress by circumventing the transient solution v_0 since this short-lived solution has negligible influence at later times. This can be achieved by enforcing the steady-state velocity, v_{s}, as the initial condition for v_0 (i.e. set $v_i = v_0(0) = v_{\text{s}}$). Doing this yields the solution $v_0(t) = v_{\text{s}}$, that is, the velocity remains constant with time. Thus, $x_0(t) = v_{\text{s}} t$ and equation (14) simplifies to

$$\dot{v}_1 + \left(\Gamma_1 + \frac{P}{v_{\text{s}}^2}\right) v_1 = -\beta y'(v_{\text{s}} t) - \mu v_{\text{s}}^2 y''(v_{\text{s}} t), \tag{15}$$

which now takes the form of a linear, non-homogeneous, first-order differential equation with constant coefficients for v_1 and can be solved once $y(x)$ is specified. Hence, the approximate solution for the velocity to first order becomes $v(t) \approx v_s + \varepsilon v_1(t)$.

The equations for the subsequent terms in the series get increasingly more complicated and difficult to solve analytically. Fortunately, as we will see in the next section, the approximate solution, $v_s + \varepsilon v_1(t)$, yields good agreement with the numerical solution of equation (11).

4. Numerical simulations

A numerical method was implemented to solve (7). To accomplish this we have adopted the fourth-order Runge–Kutta (RK4) algorithm (Recktenwald [10]) because of its simplicity, accuracy and popularity. In order to apply this technique equation (7) was expressed as the following coupled system of first-order differential equations

$$\dot{x} = G_1(w) = w,$$

$$\dot{w} = G_2(x, w) = -\frac{y'y''w^2}{[1+(y')^2]} - \frac{f_{\text{air}}}{M\sqrt{1+(y')^2}} + \frac{P}{Mw[1+(y')^2]} - \frac{gy'}{[1+(y')^2]} - \frac{\mu(g+y''w^2)}{[1+(y')^2]}.$$

When applied to this system the RK4 method advances the solution from time t_n to time $t_{n+1} = t_n + \Delta t$ according to the marching algorithm given by

$$x_{n+1} = x_n + \frac{\Delta t}{6}\left(m_1 + 2m_2 + 2m_3 + m_4\right),$$

$$w_{n+1} = w_n + \frac{\Delta t}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right),$$

where

$$m_1 = G_1(w_n), \qquad k_1 = G_2(x_n, w_n),$$

$$m_2 = G_1\left(w_n + \frac{k_1\Delta t}{2}\right), \qquad k_2 = G_2\left(x_n + \frac{m_1\Delta t}{2}, w_n + \frac{k_1\Delta t}{2}\right),$$

$$m_3 = G_1\left(w_n + \frac{k_2\Delta t}{2}\right), \qquad k_3 = G_2\left(x_n + \frac{m_2\Delta t}{2}, w_n + \frac{k_2\Delta t}{2}\right),$$

$$m_4 = G_1(w_n + k_3\Delta t), \qquad k_4 = G_2(x_n + m_3\Delta t, w_n + k_3\Delta t).$$

Here, x_n, w_n are the computed solutions at time t_n while x_{n+1}, w_{n+1} are the sought after solutions at time t_{n+1}. For the parameter values listed in table 1 a time step of $\Delta t = 0.1$ s was used. The MATLAB programme (titled 'RollingHills.m') used to solve the above system of equations is provided as supplemental material [11].

A simulation was conducted to mimic a cyclist riding over rolling hills with zero wind. The terrain was prescribed using

$$y(x) = A\,\cos\left(\frac{2\pi x}{\lambda}\right),$$

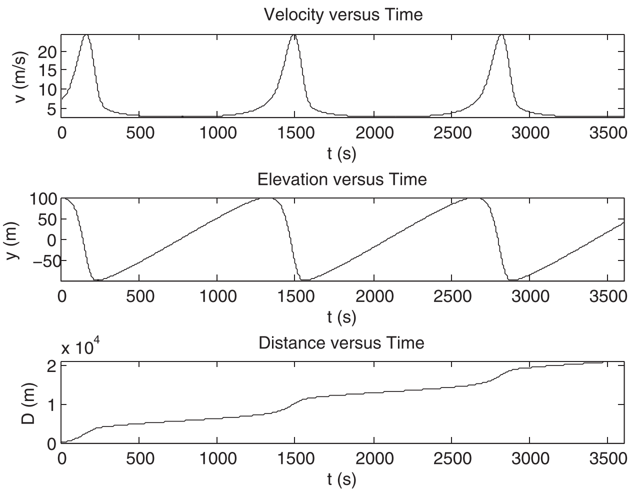

Figure 3. Velocity, elevation and distance versus time for a cyclist riding over rolling hills with air resistance.

where $A = 100$ m and $\lambda = 7,500$ m. To illustrate the impact of air resistance simulations were carried out with and without air resistance. Air resistance was incorporated using $f_{\mathrm{air}} = kv$. The cyclist's power output was held constant at $P = 250$ W. The initial velocity was taken to be the steady-state velocity v_{s} given in section 3.1 for the case with air resistance, and the steady-state velocity u_{c} given in section 3.2 with $\alpha = 0$ for the case without air resistance. To test our numerical solution procedure a comparison was made with the MATLAB inbuilt solver ode45 and the agreement was excellent.

In the following figures positions are given in metres, m, time in seconds, s, and velocities are reported in metres per second, m s^{-1}. The simulations spanned a time interval of 1 h. Shown in figure 3 are the velocity, elevation and distance travelled versus time for the case with air resistance. The corresponding plots without air resistance are displayed in figure 4. Comparing figures 3 and 4 we observe a significant increase in the maximum velocity reached, 54.8 m s^{-1} (197 km hr^{-1}) in figure 4 versus 24.6 m s^{-1} (88.6 km hr^{-1}) in figure 3, as well as a significant increase in the total distance travelled, 44.2 km in figure 4 versus 21.1 km in figure 3. We see that air resistance reduces the maximum velocity and the total distance travelled by slightly more than a factor of two. In reality, the maximum velocities would be less because one does not typically pedal while descending. Since the power was held constant at $P = 250$ W this means that the cyclist was pedalling both up and down the hills in this simulation. We also notice that there is little change in the minimum velocity for the two cases, 2.8 m s^{-1} (10.1 km hr^{-1}) in figure 4 versus 2.6 m s^{-1} (9.4 km hr^{-1}) in figure 3. This makes sense since air resistance has little effect at low velocities. In general, without air resistance the curves displayed in figure 4 can be characterized as having short intervals with rapid changes followed by longer intervals with much less variation. The curves shown in figure 3 with air resistance, on the other hand, show more gradual changes when compared to those in figure 4. Lastly, it is interesting to note that the maximum velocity occurs somewhere between the position of the maximum downhill slope and the bottom of the hill. The minimum velocity, on the other hand, occurs close to the position where the uphill slope is greatest. This is confirmed in figure 5 where the velocity and elevation are plotted versus the x position for the case with air resistance. For the landscape profile shown in figure 5 the downhill slope is greatest at $x = 1,875$ m, $x = 9,375$ m and

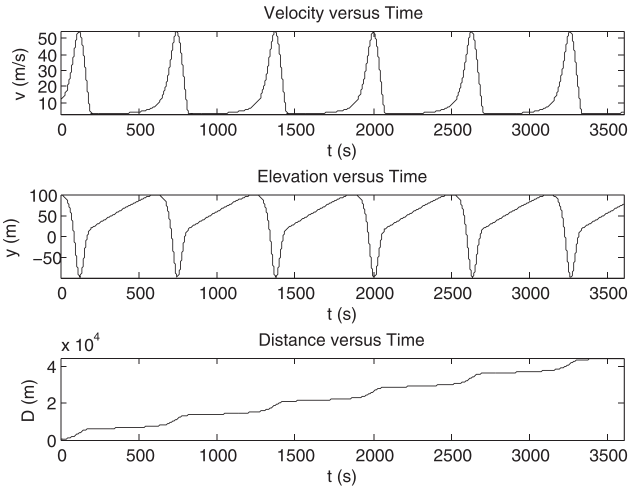

Figure 4. Velocity, elevation and distance versus time for a cyclist riding over rolling hills without air resistance.

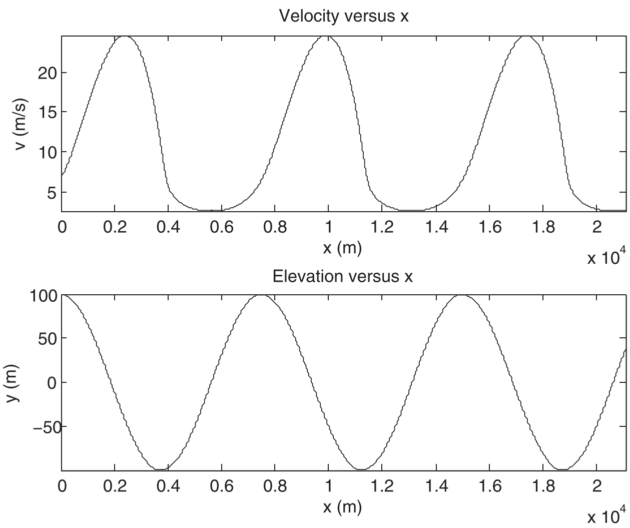

Figure 5. Velocity and elevation versus position x for a cyclist riding over rolling hills with air resistance.

$x = 16,875$ m whereas the uphill slope is greatest at $x = 5,625$ m, $x = 13,125$ m and $x = 20,625$ m. Also, the bottom of the hills occur at $x = 3,750$ m, $x = 11,250$ m and $x = 18,750$ m.

As a final note we present a comparison between the analytical solution given by (15) and the numerical solution of equation (11) for the case discussed above. In dimensionless form the rolling hills terrain is given by $y(x) = \cos(2\pi x)$ and equation (15) becomes

$$\dot{v}_1 + \left(\Gamma_1 + \frac{P}{v_\mathrm{s}^2}\right) v_1 = 2\pi\beta\, \sin(2\pi v_\mathrm{s} t) + 4\pi^2 \mu v_\mathrm{s}^2\, \cos(2\pi v_\mathrm{s} t).$$

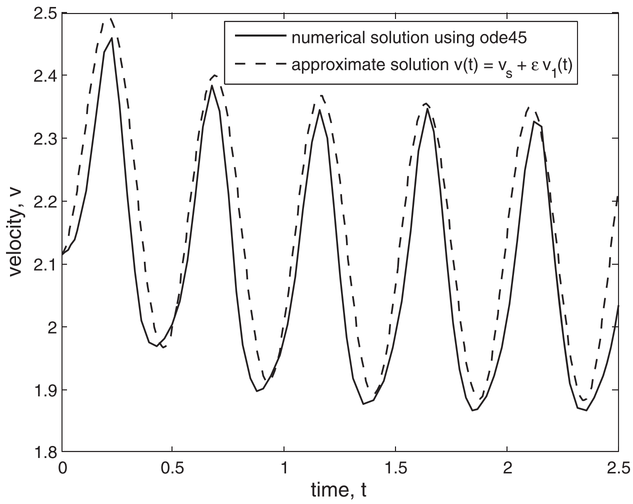

Figure 6. Comparison between the analytical and numerical solutions with $\varepsilon = 0.05$, $\beta = 10$, $\Gamma_1 = 1$, $\mu = 0.025$, $P = 5$ and $v_s = (-\mu\beta + \sqrt{4\Gamma_1 P + \mu^2\beta^2})/(2\Gamma_1)$.

The exact solution is easily found to be

$$v_1(t) = c_1 \sin(2\pi v_s t) + c_2[\cos(2\pi v_s t) - \exp(-\gamma_0 t)],$$

where

$$c_1 = \frac{2\pi(\gamma_0\beta + 4\pi^2\mu v_s^3)}{4\pi^2 v_s^2 + \gamma_0^2}, \qquad c_2 = \frac{4\pi^2 v_s(\mu\gamma_0 v_s - \beta)}{4\pi^2 v_s^2 + \gamma_0^2}, \qquad \gamma_0 = \Gamma_1 + \frac{P}{v_s^2}.$$

Plotted in figure 6 is a comparison in velocity between the approximate and numerical solutions. The numerical solution was obtained using the MATLAB solver ode45. The diagram reveals good agreement. As ε decreases the agreement will improve while as ε increases it worsens. The agreement would also improve if more terms in the series are included. The approximate solution provides insightful information that a numerical solution cannot. For example, it is able to predict mathematical expressions for the period and amplitude of the oscillatory behaviour.

5. Summary

A mathematical model describing a cyclist riding over uneven terrain has been formulated in terms of a nonlinear differential equation. The derived model is capable of handling any terrain and incorporates several parameters that can be prescribed to account for road friction, air resistance and the cyclist's power output. Although exact solutions have been found for special cases, the general case can only be solved numerically. An approximate analytical solution procedure was also proposed. This involved casting the equation in dimensionless form and identifying a small parameter. An approximate solution was then constructed in the form of an expansion in powers of this small parameter. This technique was successful in retaining the dominant terms in the equation. The goal behind an approximate analytical solution procedure is to be able to simplify the equation so as to obtain a closed form mathematical solution which not only adds insight into the problem but also agrees well with the numerical solution. The approximate solution obtained in this investigation was indeed found to be in good agreement with the numerical solution. A successful numerical solution procedure was also presented and tested against the inbuilt MATLAB solver ode45. The numerical scheme was then utilized to simulate a cyclist riding over a sinusoidally varying terrain. The simulations revealed that air resistance plays a significant role in the dynamics. In the simulations presented it reduced the maximum velocity and the total distance travelled by about a factor of two. Lastly, this study served as an example of the application of mathematical modelling and coding to a real-life problem. Although the formulation focussed on a cyclist riding along a variable landscape it could easily apply to a vehicle driving along an uneven road surface.

ORCID iDs

Serge D'Alessio https://orcid.org/0000-0002-0350-4672

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