Chasing Imaginary Triangles

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Problem: A right triangle has a perimeter of 10 and an area of 5. What is the length of its hypotenuse?

This suggests:

Question 1. Given a right triangle with a given area A and a given perimeter P, how can we find the length of its hypotenuse?

Suppose the triangle has hypotenuse c and legs a and b. Then

$$A = \tfrac{1}{2}ab, \qquad P = a+b+c, \qquad \text{and} \qquad a^2+b^2 = c^2.$$

We want to find c in terms of A and P. If we wished to use brute force, we could substitute $c = \sqrt{a^2+b^2}$ into the second equation. This would give us $a+b+\sqrt{a^2+b^2} = P$, which we could solve along with $ab = 2A$ to find a and b, and then we could substitute back to obtain c.

However, there is a more clever way:

$$\begin{aligned} P - c &= a+b, \\ P^2 - 2Pc + c^2 &= a^2+2ab+b^2 \;=\; c^2+4A, \\ P^2 - 4A &= 2Pc, \\ c &= \frac{P^2-4A}{2P}. \end{aligned}$$

Wonderful! We were able to solve for the hypotenuse c without solving for a and b (which could potentially be messy).

For mathematical convenience we introduce the semiperimeter s. Then $P = 2s$, and the above expression takes on the simpler form

$$c = \frac{s^2-A}{s}.$$

In the given problem, $c = (5^2-5)/5 = 4$.

This leads to:

Question 2. Given a right triangle with a given integral area A and a given integral perimeter P, when is the length of the hypotenuse also an integer?

For c to be an integer, we would like $2P \mid (P^2-4A)$. Since $2P$ is even, P^2-4A must be even; whence, P must be even. Since $P = 2s$, then s must be an integer, and our condition is equivalent to

$$4s \mid (4s^2-4A) \quad\Longleftrightarrow\quad s \mid (s^2-A) \quad\Longleftrightarrow\quad s \mid A.$$

Hence, any right triangle with integral area and with a semiperimeter which divides the area will have a hypotenuse with integer length.

The story could end there. But, when solving the initial problem, one of us actually tried to calculate the legs a and b. This is not so difficult now that we know the length of the hypotenuse. We have $\frac{1}{2}ab = 5$ and $a+b = 6$, which yields $a(6-a) = 10$, or $a^2-6a+10 = 0$; hence, $a = 3\pm i$. Therefore, this triangle does not actually exist!

Question 3. Given a "right triangle" with a given area A and a given perimeter P, what conditions on A and P guarantee that the "triangle" actually exists?

We go back again to our initial equations,

$$A = \tfrac{1}{2}ab \qquad \text{and} \qquad P = a+b+c.$$

Using our result for c in terms of s and A in the equation $a+b+c = P$, we get

$$a+b = 2s - \frac{s^2-A}{s} = \frac{s^2+A}{s}.$$

Solving for b and substituting into the equation $ab = 2A$, we get

$$\begin{aligned} a\left(\frac{s^2+A}{s} - a\right) &= 2A, \\ sa^2 - (s^2+A)a + 2As &= 0. \end{aligned}$$

For the triangle to exist, the discriminant must be non-negative; that is,

$$\begin{aligned} (s^2+A)^2 - 4(s)(2As) &\geq 0, \\ s^4 + 2As^2 + A^2 - 8As^2 &\geq 0, \\ s^4 - 6As^2 + A^2 &\geq 0. \end{aligned}$$

Although this is a nice result, we can go further. Dividing through by A^2 and letting $x = s^2/A$, we obtain

$$x^2 - 6x + 1 \geq 0.$$

Solving this quadratic inequality, we get

$$x \geq 3+2\sqrt{2} \qquad \text{or} \qquad x \leq 3-2\sqrt{2},$$

which is equivalent to

$$s^2 \geq (3+2\sqrt{2})A \qquad \text{or} \qquad s^2 \leq (3-2\sqrt{2})A;$$

that is,

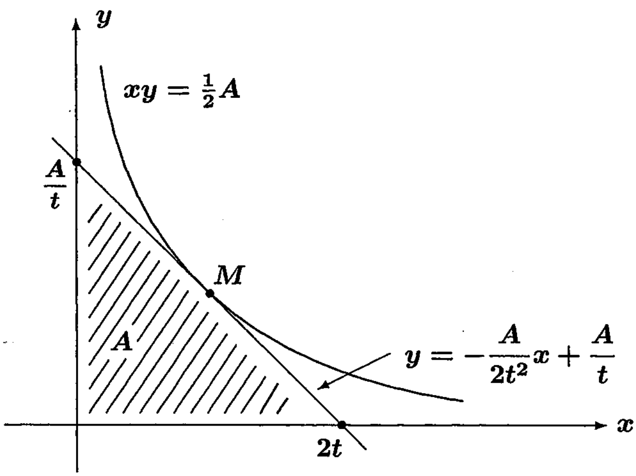
$$s \geq (1+\sqrt{2})\sqrt{A} \qquad \text{or} \qquad s \leq (\sqrt{2}-1)\sqrt{A}.$$

But we should not forget to check the hypotenuse, which will certainly be a real number (as long as A and s are real), but needs to be positive! For $c > 0$, we need $s^2 - A > 0$; that is, $s > \sqrt{A}$. Hence, the second of the above inequalities yields a triangle with real legs, but negative hypotenuse. (How did that happen?) The first inequality is consistent with $s > \sqrt{A}$.

Therefore, the "right triangle" with area A and semiperimeter s is a *bona fide* triangle if and only if $s \geq (1+\sqrt{2})\sqrt{A}$.

Returning to our original problem where $s = 5$ and $A = 5$, we find that $(1+\sqrt{2})\sqrt{A} \approx 5.4$, which clearly violates the above condition and again confirms that such a triangle does not exist.

We next proceed to offer a geometric interpretation of the condition $s \geq (1+\sqrt{2})\sqrt{A}$. Consider the hyperbola $xy = A/2$ and a line tangent to this curve at point $M\left(t, \dfrac{A}{2t}\right)$ as shown below. We can quickly determine that the slope of the tangent line is $-\dfrac{A}{2t^2}$. Thus, the tangent line has equation $y = -\dfrac{A}{2t^2}x + \dfrac{A}{t}$, with x-intercept $2t$ and y-intercept $\dfrac{A}{t}$. We observe that M is the mid-point of the portion of the tangent line that is cut off by the axes. More interesting, though, is the fact that the area of the right triangle formed by the tangent line and the coordinate axes is the same, regardless of where M is located on the hyperbola. As we let M move along the hyperbola, every possible shape of right triangle with area A will be achieved.



Now, the semiperimeter of the above triangle is given by

$$s = \frac{1}{2}\left(2t + \frac{A}{t} + \sqrt{4t^2 + \frac{A^2}{t^2}}\right) = t + \frac{A}{2t} + \sqrt{t^2 + \frac{A^2}{4t^2}}\,.$$

Since we have established that A is constant for all such triangles, we can think of this as a (differentiable) function of a single variable t, where $0 < t < \infty$. Because $s \to \infty$ as $t \to 0$ or $t \to \infty$, and since $s > 0$ for all $t \in (0, \infty)$, it is clear that s must possess an absolute minimum. The minimum value can be found by setting $\dfrac{ds}{dt} = 0$, where

$$\frac{ds}{dt} = 1 - \frac{A}{2t^2} + \frac{1 - \dfrac{A^2}{4t^4}}{\sqrt{1 + \dfrac{A^2}{4t^4}}}\,.$$

This leads to $t = \sqrt{A/2}$ and corresponds to the special case where the right triangle is isosceles. Setting $t = \sqrt{A/2}$ in our formula for s in terms of t and simplifying, we obtain $s_{\min} = (1+\sqrt{2})\sqrt{A}$, from which the condition $s \geq (1+\sqrt{2})\sqrt{A}$ follows immediately.

As a final note we wish to make a connection with Heron's formula, which relates the area A, semiperimeter s, and lengths ℓ_1, ℓ_2, ℓ_3 of the sides of an arbitrary triangle according to the expression

$$A = \sqrt{s(s-\ell_1)(s-\ell_2)(s-\ell_3)}\,.$$

In our original problem with $A = 5$ and $s = P/2 = 5$, the above simplifies to $5 = (5-\ell_1)(5-\ell_2)(5-\ell_3)$. Setting $X = 5-\ell_1$, $Y = 5-\ell_2$, and $Z = 5-\ell_3$, it follows that $X+Y+Z = 5$, since $l_1+l_2+l_3 = P = 10$.

Thus, we have $XYZ = 5$ and $\frac{1}{3}(X+Y+Z) = \frac{5}{3}$. Enforcing the AM–GM Inequality yields

$$XYZ \leq \left(\frac{X+Y+Z}{3}\right)^3,$$

which leads to $5 \leq 125/27$, a contradiction. Hence, such a triangle cannot exist. This reaffirms our earlier finding.

Furthermore, using Heron's formula again for a general triangle, we have

$$A^2/s = (s-\ell_1)(s-\ell_2)(s-\ell_3)\,.$$

If we set $X = s-\ell_1$, $Y = s-\ell_2$, and $Z = s-\ell_3$, then we see that

$$X+Y+Z = s \qquad \text{and} \qquad XYZ = A^2/s\,.$$

Again, by the AM–GM Inequality, it follows that a necessary condition for such a triangle to exist is

$$\frac{A^2}{s} \leq \left(\frac{s}{3}\right)^3$$

(or $s^2 \geq 3\sqrt{3}A$); equality occurs if and only if the triangle is equilateral. Lastly, we point out that this condition applies to all triangles, whereas the previously derived condition, $s \geq \left(1+\sqrt{2}\right)\sqrt{A}$, applies only to right triangles.

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