Faculty of Mathematics

Centre for Education in Mathematics and Computing

CHASING IMAGINARY TRIANGLES

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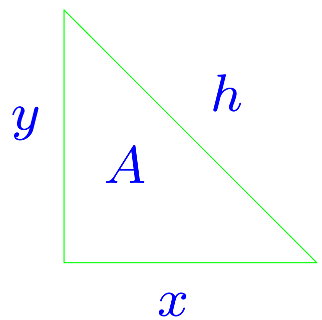
Problem

A right-angled triangle has a perimeter (P) of 10 and an area (A) of 5. What is the length of its hypotenuse (h)?



$$P = x + y + h \; , \; A = \frac{1}{2}xy \; , \; h = \sqrt{x^2 + y^2}$$

Question 1

Given a right-angled triangle with given area A and given perimeter P, how can we easily find its hypotenuse?

Solution: From the diagram we have

$$A = \frac{1}{2}xy \, , \; P = x + y + h \, , \; h^2 = x^2 + y^2 \, .$$

Substituting $h = \sqrt{x^2 + y^2}$ into the second equation gives

$$\begin{aligned} xy &= 2A \, , \\ x + y + \sqrt{x^2 + y^2} &= P \end{aligned}$$

which we can solve for x and y and then substitute back to obtain h.

Alternate Solution:

Proceed as follows:

$$\begin{aligned} P - h &= x + y \\ (P-h)^2 &= (x+y)^2 \\ P^2 - 2Ph + h^2 &= x^2 + 2xy + y^2 \\ P^2 - 2Ph + h^2 &= h^2 + 4A \\ P^2 - 4A &= 2Ph \\ h &= \frac{P^2 - 4A}{2P} \end{aligned}$$

In the given example, $h = \dfrac{10^2 - 4(5)}{2(10)} = 4$. This method avoids solving for the lengths of the sides of the triangle (which could be messy).

Question 2

Given a right-angled triangle with given integral area A and given integral perimeter P, when is the length of the hypotenuse also an integer?

Solution: For h to be an integer, we require $2P \mid (P^2 - 4A)$. Since $2P$ is even, then $P^2 - 4A$ must be even, so P must be even. Setting $P = 2p$ (so p is also an integer) our condition becomes

$$4p \mid (4p^2 - 4A) \iff p \mid (p^2 - A) \iff p \mid A \iff \frac{1}{2}P \mid A \iff P \mid 2A$$

Thus, any right-angled triangle where the perimeter divides twice the area will have a hypotenuse with integer length. In our problem, $P = 10$ and $A = 5$ so the above condition is satisfied.

However,

when solving the initial problem by calculating the sides of the triangle x and y, an interesting result emerges. Using $h = 4$, as previously calculated, x and y must satisfy

$$\frac{1}{2}xy = 5\,,\ x + y = 6$$

which gives $x(6 - x) = 10$ or $x^2 - 6x + 10 = 0$. Solving yields $x = 3 \pm i$. So this triangle doesn't actually exist!

What went wrong?!

Question 3

Given a "right-angled triangle" with given area A and given perimeter P, what conditions on A and P guarantee that the "triangle" actually exists?

Solution: Recalling our initial equations

$$A = \frac{1}{2}xy \; , \; P = x + y + h$$

and using our result for h in terms of P and A we obtain

$$xy = 2A \; , \; x + y = P - \frac{P^2 - 4A}{2P} = \frac{P^2 + 4A}{2P} \; .$$

Solving the second equation for y and substituting, gives

$$\begin{aligned} x\left(\frac{P^2+4A}{2P}-x\right) &= 2A \\ 2Px^2-(P^2+4A)x+4AP &= 0 \end{aligned}$$

Thus, for the triangle to actually exist, the discriminant must be non-negative, i.e.

$$\begin{aligned} (P^2+4A)^2-4(2P)(4AP) &\geq 0 \\ \Leftrightarrow P^4+8AP^2+16A^2-32AP^2 &\geq 0 \\ \Leftrightarrow P^4-24AP^2+16A^2 &\geq 0 \end{aligned}$$

Aside: Using our original numbers,

$$P^4-24AP^2+16A^2=10^4-24(5)(10^2)+16(5^2)=-1600<0$$

which confirms our finding that the triangle does not exist.

We can manipulate this inequality further, although

$$P^4 - 24AP^2 + 16A^2 \geq 0$$

is a nice result. Dividing through by A^2, we obtain

$$\left(\frac{P^2}{A}\right)^2 - 24\left(\frac{P^2}{A}\right) + 16 \geq 0\,.$$

Solving this quadratic inequality, we obtain

$$\frac{P^2}{A} \geq 12 + 8\sqrt{2} \qquad \text{or} \qquad \frac{P^2}{A} \leq 12 - 8\sqrt{2}$$

which is equivalent to

$$P^2 \geq (12 + 8\sqrt{2})A \qquad \text{or} \qquad P^2 \leq (12 - 8\sqrt{2})A$$

or

$$P \geq (2 + 2\sqrt{2})\sqrt{A} \qquad \text{or} \qquad P \leq (2\sqrt{2} - 2)\sqrt{A}\,.$$

Imposing the constraint $h > 0$, requires $P^2 - 4A > 0$ or $P > 2\sqrt{A}$. Therefore, the second of the inequalities yields a triangle with real sides, but negative hypotenuse. *How did that happen?*

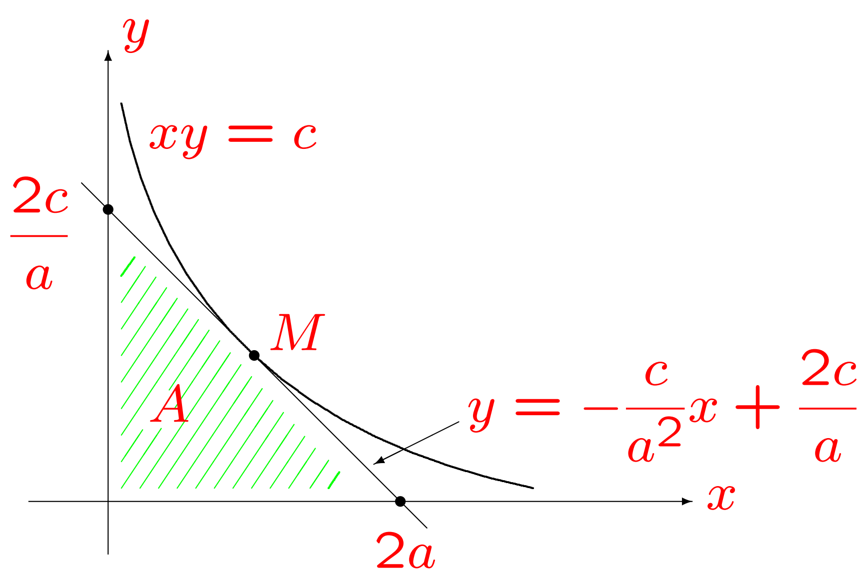
The first inequality is consistent with $P > 2\sqrt{A}$.

Thus, the "right-angled triangle" with area A and P is a bona-fide triangle if and only if $P \geq (2 + 2\sqrt{2})\sqrt{A}$.

Aside: Returning to our original triangle having $P = 10$ and $A = 5$, we find that $(2 + 2\sqrt{2})\sqrt{A} \approx 10.8$, which clearly violates the above condition and again confirms that such a triangle does not exist.

We next proceed to offer a geometric interpretation of the condition $P \geq (2+2\sqrt{2})\sqrt{A}$.

Consider the hyperbola $xy = c$ and a line tangent to this curve at point $M\left(a, \dfrac{c}{a}\right)$ as shown below.



We can easily determine that the slope of the tangent line to be $-\frac{c}{a^2}$ and the equation of the tangent line to be $y = -\frac{c}{a^2}x + \frac{2c}{a}$. We observe that M is the midpoint of the x-intercept ($x = 2a$) and the y-intercept ($y = \frac{2c}{a}$) of the tangent line. More interesting, though, is the fact that:

the area of the right-angled triangle formed by the tangent line and the coordinate axes is the same, regardless of where M is located on the hyperbola.

In particular, $A = 2c$. As we let M move along the hyperbola, every possible shape of triangle with area $A = 2c$ will be achieved.

Now, the perimeter of the above-mentioned triangle is given by

$$P = 2a + \frac{2c}{a} + \sqrt{4a^2 + \frac{4c^2}{a^2}}\,.$$

Substituting $c = \frac{1}{2}A$ into this expression yields

$$P = 2a + \frac{A}{a} + \sqrt{4a^2 + \frac{A^2}{a^2}}\,.$$

Since we have established that A is constant for all triangles, we can think of this as a (differentiable) function of a single variable a, where $0 < a < \infty$. Because $P \to \infty$ as $a \to 0, \infty$ and $P > 0$ for all $a \in (0, \infty)$, it is clear that P must possess an absolute minimum. (*Aside:* Strangely enough, as $a \to 0, \infty$ the perimeter grows without bound, yet the area remains constant!)

The minimum value can be found by setting $\frac{dP}{da} = 0$, where

$$\frac{dP}{da} = 2 - \frac{A}{a^2} + \frac{4 - \frac{A^2}{a^4}}{\sqrt{4 + \frac{A^2}{a^4}}}$$

This leads to $a = \sqrt{\frac{A}{2}}$.

Thus, $P_{min} = P\left(a = \sqrt{\frac{A}{2}}\right) = (2 + 2\sqrt{2})\sqrt{A}$, from which the condition $P \geq (2 + 2\sqrt{2})\sqrt{A}$ immediately follows.

Thank You!