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Solutions of the Van der Pol Equation

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The Van der Pol equation is one of the most extensively studied differential equations with applications ranging from the physical to the biological sciences. Owing to its cubic nonlinearity, an exact solution is still out of reach. However, various approximate solutions have been advanced over the years. Here, we highlight some of those solutions, discuss their limitations, and propose a new solution. This work also illustrates the interplay between theory and computation.

Background

In 1926, the Dutch electrical engineer Balthasar van der Pol (1889–1959) introduced the, now famous, differential equation (DE) which bears his name

$$\frac{d^2x}{dt^2} + \mu(x^2-1)\frac{dx}{dt} + x = 0 \text{ satisfying } x(0) = \alpha, \left.\frac{dx}{dt}\right|_0 = \beta, \tag{1}$$

to model triode oscillations in electrical circuits [10]. This equation describes a nonconservative oscillator having a linear spring force given by x and a nonlinear damping force represented by

$$\mu(x^2-1)\frac{dx}{dt},$$

where the parameter μ is a positive scalar which measures the strength of the damping term, x is the position and t denotes the time. We see that the sign of the damping force depends on whether $|x|$ is larger or smaller than unity. When $|x| > 1$ we have true damping, meaning that it opposes motion causing it to decay with time. However, when $|x| < 1$ the damping term has the opposite effect, that is, it amplifies the motion. At instants when $|x| = 1$ the damping force momentarily vanishes.

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The Van der Pol equation is related to other equations. For example, the Rayleigh equation

$$\frac{d^2z}{dt^2} + \mu\left(\frac{1}{3}\left[\frac{dz}{dt}\right]^2 - 1\right)\frac{dz}{dt} + z = 0,$$

can be converted to (1) through the substitution

$$z(t) = \int_0^t x(\tau)d\tau.$$

In addition, by introducing $u = \dfrac{dx}{dt}$ and writing $\dfrac{d^2x}{dt^2} = u\dfrac{du}{dx}$, equation (1) becomes

$$u\frac{du}{dx} = -\mu(x^2 - 1)u - x,$$

which takes the form of Abel's equation of the second kind [7]. Since the Rayleigh and Abel equations do not admit exact analytical solutions in terms of known elementary functions, we expect that same fate to follow for the Van der Pol equation [6]. It is worth noting, however, that an exact solution to the Duffing–Van der Pol equation given by

$$\frac{d^2x}{dt^2} + c_0(x^2 - c_1)\frac{dx}{dt} + c_2x + c_3x^3 = 0,$$

where c_0, c_1, c_2, c_3 are constants has been found [9]. Equation (1) can be studied using the framework of dynamical systems, and in this context it is well known that all nontrivial solutions tend to a unique periodic limit cycle as $t \to \infty$ which depend only on the value of the parameter $\mu(> 0)$ [5].

The absence of an exact solution has motivated researchers to construct approximate analytical solutions using well established techniques. We next present a few of them. Here, we will only list the results as our focus is on the solution presented in the following section. The details surrounding the techniques utilized to obtain the solutions are fully explained in [2]. The first takes the form of a regular perturbation solution in the parameter μ. For small μ the following solution can be found in powers of μ

$$x(t) = x_0(t) + \mu x_1(t) + \mu^2 x_2(t) + \cdots .$$

The leading and first-order solutions, x_0 and x_1, respectively, have been found and are given by

$$x_0(t) = \alpha\cos(t) + \beta\sin(t),$$

$$x_1(t) = c_1\sin(t) + c_2\cos(t) + c_3[\beta t\sin(t) + \alpha t\cos(t)] + c_4\sin(3t) + c_5\cos(3t),$$

where

$$c_1 = -\frac{\alpha}{32}(16 - 7\alpha^2 + 5\beta^2),\ c_2 = -\frac{\beta}{32}(3\alpha^2 - \beta^2),\ c_3 = \frac{1}{2}\left[1 - \frac{1}{4}\left(\alpha^2 + \beta^2\right)\right],$$

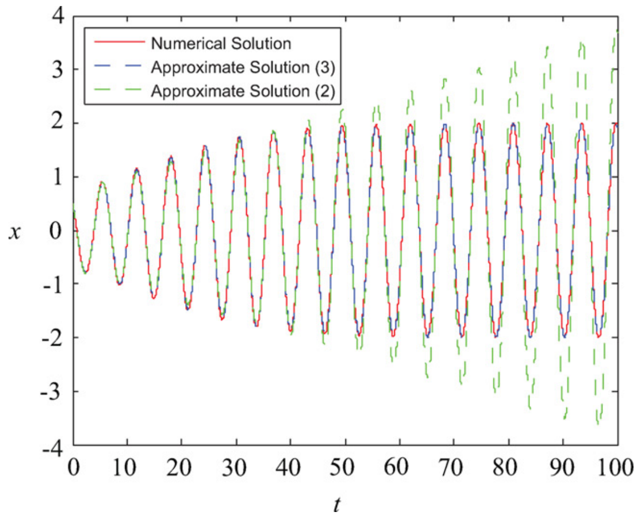

Figure 1. Comparison between the approximate solutions given by equations (2), (3) and the numerical solution.

$$c_4 = \frac{\alpha}{32}(3\beta^2 - \alpha^2), \; c_5 = \frac{\beta}{32}(3\alpha^2 - \beta^2).$$

This yields the approximate solution

$$x(t) \approx x_0(t) + \mu x_1(t). \tag{2}$$

The above approach can be continued to obtain higher-order terms [**4**]. We notice that $x_0(t)$ corresponds to the solution of (1) with $\mu = 0$. As indicated by the terms $t\sin(t)$ and $t\cos(t)$, the solution grows without bound, and the consequence of this is illustrated in Figure 1.

Another approximate solution can be obtained using the method of multiple scales [**3**]. This method tackles equations possessing multiple time scales by introducing slow and fast time variables. Using this technique the following leading-order solution can be derived

$$x(t) \approx \frac{2[\alpha\cos(t) + \beta\sin(t)]}{\sqrt{(\alpha^2+\beta^2) + (4 - \alpha^2 - \beta^2)\exp(-\mu t)}}. \tag{3}$$

Contrasted in Figure 1 are the approximate and numerical solutions for the case $\alpha = 0.5$, $\beta = -0.5$ and $\mu = 0.1$. Shown here are the approximate solutions given by (2) and (3). We see that all three solutions are in good agreement up to $t \approx 40$. After that the approximate solution given by (2) continues to grow, and noticeable departures from the numerical solution are apparent, while the approximate solution given by (3) remains in close agreement with the numerical solution over the entire time interval. Further, as $t \to \infty$ equation (3) also predicts the correct limit cycle for small μ which corresponds to the circle in the phase plane given by

$$x^2 + \left(\frac{dx}{dt}\right)^2 = 4,$$

having a period $T = 2\pi$. Although the agreement is impressive, one must remember that the agreement worsens as μ increases. The numerical solution procedure is outlined in the Appendix.

The above approximate solutions are valid for small μ. The next method is well suited to address the case of large μ. If μ is large, we can treat $1/\mu$ as a small parameter. Scaling the time according to $t = \mu\tau$ equation (1) becomes

$$\varepsilon\frac{d^2x}{d\tau^2} + (x^2 - 1)\frac{dx}{d\tau} + x = 0 \;\text{ where }\; \varepsilon = \frac{1}{\mu^2}.$$

An approximate solution to this equation can be constructed using the singular perturbation method known as matched asymptotic expansions. This technique involves splitting the domain into different regions and deriving solutions in each region and then blending them together using asymptotic matching. The solution obtained using this method is rather complicated and not worth presenting, but the details can be found in [1]. However, a key result worth sharing from this analysis is the approximate period of the limit cycle for large μ given by

$$T \approx (3 - 2\ln 2)\mu + 2\gamma\mu^{-\frac{1}{3}} \;\text{ where }\; \gamma \approx 2.338.$$

Power series solution

The solutions presented in the previous section are based on assumptions on the parameter μ. The approach outlined here makes no assumptions on the parameter μ. Instead, we will assume a power series solution having the form

$$x(t) = \sum_{n=0}^{\infty} a_n t^n, \tag{4}$$

since the initial data is prescribed at $t = 0$. Substituting this series into (1) leads to a recurrence relation for the coefficients a_n. The nonlinear term $x^2\dfrac{dx}{dt}$ can be computed using the Cauchy product of two infinite series as follows

$$x^2 = \sum_{n=0}^{\infty} b_n t^n \;\text{ with }\; b_n = \sum_{j=0}^{n} a_j a_{n-j} \;\text{ and }\; x^2\frac{dx}{dt} = \sum_{n=0}^{\infty} c_n t^n,$$

where

$$c_n = \sum_{k=0}^{n}(n-k+1)a_{n-k+1}b_k = \sum_{k=0}^{n}(n-k+1)a_{n-k+1}\sum_{j=0}^{k} a_j a_{k-j}.$$

The recurrence relation then becomes

$$a_{n+2} = \frac{\mu a_{n+1}}{(n+2)} - \frac{(a_n + \mu c_n)}{(n+2)(n+1)}, \; n = 0, 1, 2, \ldots, \tag{5}$$

and can be viewed as a nonlinear finite difference equation. It follows from the initial conditions that $a_0 = \alpha$ and $a_1 = \beta$. Although the power series given by (4) represents an exact solution to equation (1), there is an important limitation that must be remembered. Even though the coefficients a_n can be determined recursively from (5), an exact expression for a_n in terms of n and μ is likely out of reach, and thus, makes it challenging to determine the interval of convergence of (4).

To check our power series solution we consider some special cases where equation (5) can be solved exactly. If $\mu = 0$, then (5) simplifies to

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)}, \quad n = 0, 1, 2, \ldots,$$

and the solution can be divided into even and odd terms as follows

$$a_{2n} = \alpha\frac{(-1)^n}{(2n)!}, \quad a_{2n+1} = \beta\frac{(-1)^n}{(2n+1)!}, \quad n = 0, 1, 2, \ldots,$$

yielding the expected solution

$$x(t) = \alpha\cos(t) + \beta\sin(t).$$

As a second case, we ignore the term c_n and for mathematical convenience set $\mu = 2$, then (5) becomes

$$a_{n+2} = \frac{2a_{n+1}}{(n+2)} - \frac{a_n}{(n+2)(n+1)}, \quad n = 0, 1, 2, \ldots,$$

and the solution is

$$a_n = \frac{\alpha}{n!} + \frac{(\beta-\alpha)}{(n-1)!}, \quad n \geq 1 \ \text{ with } \ a_0 = \alpha.$$

Substituting this into the power series (4) we obtain

$$x(t) = [\alpha + (\beta-\alpha)t]\exp(t),$$

after some straightforward manipulations. Again, this solution is to be expected since setting $c_n = 0$ and $\mu = 2$ is equivalent to solving the DE

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = 0 \ \text{ satisfying } x(0) = \alpha, \ \left.\frac{dx}{dt}\right|_0 = \beta.$$

The special cases above illustrate that ignoring the nonlinear term yields a power series solution that is valid for all t. To shed some light on how the nonlinear term impacts the interval of convergence, we consider the DE

$$\frac{d^2x}{dt^2} = -\mu x^2\frac{dx}{dt} \ \text{ satisfying } x(0) = \alpha, \ \left.\frac{dx}{dt}\right|_0 = \beta, \tag{6}$$

which retains the highest-order derivative and the nonlinear term in (1). In terms of $u = \dfrac{dx}{dt}$, equation (6) can be written as

$$u\frac{du}{dx} = -\mu x^2 u.$$

The nontrivial solution to the above satisfying the condition $u(\alpha) = \beta$ is given by

$$u = \frac{\mu}{3}\left(C - x^3\right) \quad \text{where} \quad C = \alpha^3 + \frac{3\beta}{\mu}.$$

While this equation can be solved for $x(t)$ for arbitrary C, we will explore the case when $C = 0$ more closely. In this case the initial conditions are related through the relation $\beta = -\mu\alpha^3/3$ and the solution to the above equation is easily found to be

$$x(t) = \pm\alpha\left(1 + \frac{2}{3}\mu\alpha^2 t\right)^{-\frac{1}{2}}.$$

Using the Binomial series the above can be written as

$$x(t) = \pm\alpha \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!\mu^n \alpha^{2n}}{6^n (n!)^2} t^n \quad \text{for} \quad \frac{2}{3}\mu\alpha^2 |t| < 1.$$

Since we are only interested in the solution for $t > 0$, the above series solution is valid for $t < \dfrac{3}{2\mu\alpha^2}$. Thus, the solution to (6) can be expressed as the power series

$$x(t) = \sum_{n=0}^{\infty} \hat{a}_n t^n \quad \text{with } \hat{a}_n = \frac{(-1)^n (2n)!\mu^n \alpha^{2n+1}}{6^n (n!)^2} \quad \text{for} \quad t < \frac{3}{2\mu\alpha^2}.$$

Without loss of generality, we have taken the positive root.

As a curiosity, we ran some numerical experiments to determine if the interval of convergence given by $t < \dfrac{3}{2\mu\alpha^2}$ also applies to the Van der Pol equation (1) for the special case when the initial conditions satisfy $\beta = -\mu\alpha^3/3$. Shown in Figure 2 is a comparison between the power series solution (4), the approximate solution given by equation (3) and the numerical solution for the case $\alpha = \mu = 1$ and $\beta = -\mu\alpha^3/3 = -1/3$. We see excellent agreement between the power series and numerical solutions for $t \leq 1.5$. For $t > 1.5$ the powers series diverged. Hence, the interval of convergence agrees well with $t < \dfrac{3}{2\mu\alpha^2} = 1.5$. In computing the power series solution only 25 terms were needed as the series converged rapidly within the interval of convergence. The poor agreement between equation (3) and the numerical solution is to be expected since μ is no longer small. Several numerical experiments were carried out with different values of μ and α and the interval of convergence continued to agree well with $t < \dfrac{3}{2\mu\alpha^2}$.

To our knowledge the proposed power series solution has never appeared in the literature. It can be interpreted as an alternate solution, in addition to the various approximate solutions that have been advanced over the years. The advantage of this solution is that it represents an exact solution to the Van der Pol equation over the interval of convergence. The disadvantage is that the interval of convergence is, in general, unknown. By investigating the special case when the initial conditions satisfy $\beta = -\mu\alpha^3/3$ the interval of convergence has been estimated to be $t < \dfrac{3}{2\mu\alpha^2}$ and this was confirmed numerically. As displayed in Figure 2, the power series solution can provide a solution over intervals where other approximate solutions are not valid.

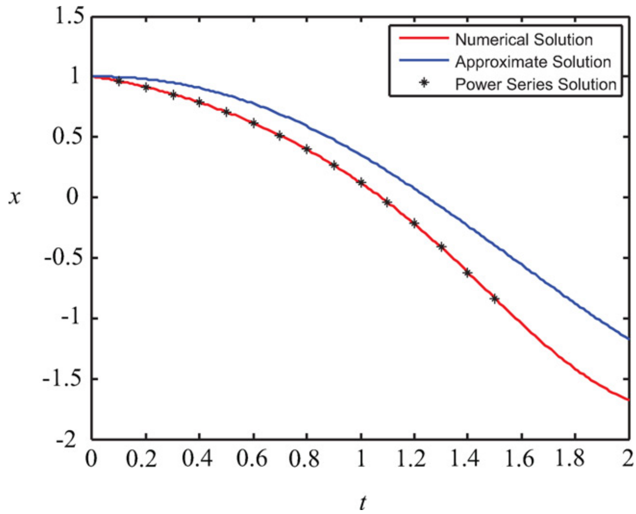

Figure 2. Comparison between the power series solution, approximate solution given by equation (3) and the numerical solution.

As a concluding remark we note that the power series solution also provides a recursive formula for the nth derivative of the Van der Pol solution evaluated at $t = 0$. By comparing (4) with its Maclaurin series it immediately follows that

$$\left.\frac{d^n x}{dt^n}\right|_0 = n!a_n \quad \text{for} \quad n = 0, 1, 2, \ldots,$$

with a_n satisfying (5). Lastly, we can also prove uniqueness of the Maclaurin series as follows. Let v_1 and v_2 denote two distinct solutions to (1), then

$$\frac{d^2 v_1}{dt^2} + \mu(v_1^2 - 1)\frac{dv_1}{dt} + v_1 = 0 \;\; \text{satisfying } v_1(0) = \alpha, \;\left.\frac{dv_1}{dt}\right|_0 = \beta,$$
$$\frac{d^2 v_2}{dt^2} + \mu(v_2^2 - 1)\frac{dv_2}{dt} + v_2 = 0 \;\; \text{satisfying } v_2(0) = \alpha, \;\left.\frac{dv_2}{dt}\right|_0 = \beta.$$

Subtracting these equations and defining $w = v_1 - v_2$ we obtain

$$\frac{d^2 w}{dt^2} - \mu\frac{dw}{dt} + w + \frac{\mu}{3}\frac{d(\xi w)}{dt} = 0 \;\; \text{satisfying } w(0) = 0, \;\left.\frac{dw}{dt}\right|_0 = 0, \tag{7}$$

where $\xi = v_1^2 + v_1 v_2 + v_2^2$. Here, we have used

$$v_1^2\frac{dv_1}{dt} - v_2^2\frac{dv_2}{dt} = \frac{1}{3}\frac{d}{dt}\left(v_1^3 - v_2^3\right) = \frac{1}{3}\frac{d}{dt}\left([v_1 - v_2][v_1^2 + v_1 v_2 + v_2^2]\right) = \frac{1}{3}\frac{d(\xi w)}{dt}.$$

Since $w(0) = 0$ and $\left.\frac{dw}{dt}\right|_0 = 0$ it follows from (7) that $\left.\frac{d^2 w}{dt^2}\right|_0 = 0$ and repeated differentiation of (7) yields $\left.\frac{d^n w}{dt^n}\right|_0 = 0$ for $n = 3, 4, 5, \ldots$. Since all the derivatives vanish, we have that $w(t) = 0$ and the desired result $v_1 = v_2$ immediately follows.

Appendix

To numerically solve equation (1) the fourth-order Runge-Kutta (RK4) algorithm [8] was adopted because of its simplicity, accuracy and popularity. In order to apply this technique equation (1) was expressed as the following coupled system of first-order differential equations

$$\frac{dx}{dt} = G_1(u) = u,$$

$$\frac{du}{dt} = G_2(x, u) = -x - \mu(x^2 - 1)u.$$

When applied to this system the RK4 method advances the solution from time t_n to time $t_{n+1} = t_n + \Delta t$ according to the marching algorithm given by

$$x_{n+1} = x_n + \frac{\Delta t}{6}\left(m_1 + 2m_2 + 2m_3 + m_4\right),$$

$$u_{n+1} = u_n + \frac{\Delta t}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right),$$

where

$$m_1 = G_1(u_n),\ k_1 = G_2(x_n, u_n),$$

$$m_2 = G_1\left(u_n + \frac{k_1 \Delta t}{2}\right),\ k_2 = G_2\left(x_n + \frac{m_1 \Delta t}{2}, u_n + \frac{k_1 \Delta t}{2}\right),$$

$$m_3 = G_1\left(u_n + \frac{k_2 \Delta t}{2}\right),\ k_3 = G_2\left(x_n + \frac{m_2 \Delta t}{2}, u_n + \frac{k_2 \Delta t}{2}\right),$$

$$m_4 = G_1(u_n + k_3 \Delta t),\ k_4 = G_2(x_n + m_3 \Delta t, u_n + k_3 \Delta t).$$

Here, x_n, u_n are the computed solutions at time t_n while x_{n+1}, u_{n+1} are the sought after solutions at time t_{n+1}. To test the numerical solution procedure a comparison was made with the MATLAB inbuilt solver **ode45** and the agreement was excellent. In our simulations a time step of $\Delta t = 0.01$ was used. The MATLAB program (titled "VanderPol.m") used to solve the above system of equations is provided as supplemental material.

Summary. Presented in this paper are various solutions to the Van der Pol equation. Numerical solutions are utilized as an independent means of validating the various solutions discussed. A new solution in the form of a power series has been found. Although this solution is exact, its interval of convergence can only be estimated for a special case. Numerical experiments reveal that the power series solution can provide an exact solution over intervals where other approximate solutions are not valid. Thus, the new solution represents an additional solution that can complement other existing solutions. This work also emphasizes the importance and the role of computation. Although the power series solution along with the other approximate solutions mentioned are of theoretical interest, their restrictions limit their usefulness in real applications, and therefore numerical methods should also be considered.

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