A Numerical Method for Studying Impulsively Generated Convection from Heated Tubes

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Flow Configuration

Fluid Properties:
- $\nu$ - kinematic viscosity
- $\kappa$ - thermal diffusivity
- $k$ - thermal conductivity
- $\alpha$ - thermal expansion coefficient

Equation of State:
$$\rho = \rho_\infty [1 - \alpha(T - T_\infty)]$$

Dimensionless Parameters:
$$Gr = \frac{\alpha g \Delta T c^3}{\nu^2} \quad \text{where} \quad c = \sqrt{a^2 - b^2}$$
and
$$\Delta T = \frac{Q c}{k}, \quad Pr = \frac{\nu}{\kappa}, \eta, r$$
Unsteady free convection from a heated tube is a fundamental problem and is of interest for theoretical and practical reasons. Applications include:

- hot wire anemometry
- thermal pollution
- design of heat exchangers
The present study differs from previous investigations in the following ways:

- extend previous results on circular cylinders ([1])
- propose a new robust numerical method designed to capture the known physical behaviour
- offer an analytical solution procedure useful for theoretical and validation purposes
Figure 2.1: The conformal transformation

\[ x + iy = \cosh[(\xi + \xi_0) + i\theta] \]

where \( \tanh \xi_0 = r \), and \( r = \frac{a}{b} \) is the ratio of the semi-minor to semi-major axes of the ellipse.

This choice of the constant \( \xi_0 \) is such that the contour \( \xi = 0 \) will coincide with the surface of the cylinder. In terms of the coordinates \((\xi, \theta)\), the domain is confined to the semi-infinite rectangular strip \( \xi \geq 0, \quad 0 \leq \theta \leq 2\pi \), (see figure 2.1). In the above figure, \( \theta = 0 \) and \( \theta = \pi \) correspond to the leading and trailing tips of the cylinder respectively.

This transformation has been used in several other works, including D’Alessio \[4\] and Saunders \[15\].

Recalling that:

\[ \cosh x = e^x + e^{-x} \]
Navier-Stokes & Temperature Equations

The dimensionless unsteady equations for a viscous, incompressible fluid in terms of the streamfunction, $\psi$, vorticity, $\zeta$, and temperature, $\phi$, are:

\[
\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \theta^2} = M^2 \zeta
\]

\[
\frac{\partial \zeta}{\partial t} = \frac{1}{M^2} \left[ \frac{\partial \psi}{\partial \theta} \frac{\partial \zeta}{\partial \xi} - \frac{\partial \psi}{\partial \xi} \frac{\partial \zeta}{\partial \theta} + \frac{1}{\sqrt{Gr}} \left( \frac{\partial^2 \zeta}{\partial \xi^2} + \frac{\partial^2 \zeta}{\partial \theta^2} \right) + A \frac{\partial \phi}{\partial \xi} - B \frac{\partial \phi}{\partial \theta} \right]
\]

\[
\frac{\partial \phi}{\partial t} = \frac{1}{M^2} \left[ \frac{\partial \psi}{\partial \theta} \frac{\partial \phi}{\partial \xi} - \frac{\partial \psi}{\partial \xi} \frac{\partial \phi}{\partial \theta} + \frac{1}{\sqrt{GrPr}} \left( \frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial \theta^2} \right) \right]
\]

where functions $M, A, B$ are related to the geometry.
Surface conditions include no-slip and constant heat flux:

\[ \psi = \frac{\partial \psi}{\partial \xi} = 0 \quad \text{and} \quad \frac{1}{M} \frac{\partial \phi}{\partial \xi} = -1 \quad \text{on} \quad \xi = 0 \]

Far-field conditions: \( \psi, \zeta, \phi \to 0 \) as \( \xi \to \infty \)
Initial conditions: \( \psi = \zeta = \phi = 0 \) at \( t = 0 \)

The vorticity can be shown to satisfy global conditions:

\[
\int_{0}^{\infty} \int_{0}^{2\pi} e^{-n\xi} M^2 \zeta \sin(n\theta) d\theta d\xi = 0, \quad n = 1, 2, \ldots
\]

\[
\int_{0}^{\infty} \int_{0}^{2\pi} e^{-n\xi} M^2 \zeta \cos(n\theta) d\theta d\xi = 0, \quad n = 0, 1, \ldots
\]
Boundary Layer Transformation

Introduce boundary-layer coordinate: \( \xi = \lambda z \), \( \lambda = \sqrt{\frac{4t}{\sqrt{Gr}}} \)

The grid expands with time as illustrated below:
Boundary Layer Transformation

The governing equations then become:

\[ \frac{\partial^2 \psi}{\partial z^2} + \lambda^2 \frac{\partial^2 \psi}{\partial \theta^2} = \lambda^2 M^2 \zeta \]

\[ \frac{1}{M^2} \frac{\partial^2 \zeta}{\partial z^2} + 2z \frac{\partial \zeta}{\partial z} = 4t \frac{\partial \zeta}{\partial t} - \frac{\lambda^2}{M^2} \frac{\partial^2 \zeta}{\partial \theta^2} \]

\[ + \frac{4t}{\lambda M^2} \left( \frac{\partial \psi}{\partial z} \frac{\partial \zeta}{\partial \theta} - \frac{\partial \psi}{\partial \theta} \frac{\partial \zeta}{\partial z} \right) - \frac{4tA}{\lambda M^2} \frac{\partial \phi}{\partial z} + \frac{4tB}{M^2} \frac{\partial \phi}{\partial \theta} \]

\[ \frac{1}{PrM^2} \frac{\partial^2 \phi}{\partial z^2} + 2z \frac{\partial \phi}{\partial z} = 4t \frac{\partial \phi}{\partial t} - \frac{\lambda^2}{Pr M^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{4t}{\lambda M^2} \left( \frac{\partial \psi}{\partial z} \frac{\partial \phi}{\partial \theta} - \frac{\partial \psi}{\partial \theta} \frac{\partial \phi}{\partial z} \right) \]
Discretization

Early stages of the flow are computed using the boundary-layer coordinate $z$. Once the boundary layer thickens the flow is computed using the original coordinate $\xi$. For large $Gr$ it is more practical to work entirely in the coordinate $z$. The computational domain bounded by $0 \leq z \leq z_\infty$ and $0 \leq \theta \leq 2\pi$ is discretized into a uniform network of $K \times L$ grid points located at

$$z_i = ih_z \ , \ i = 0, 1, \ldots, K \ , \ h_z = \frac{z_\infty}{K}$$

$$\theta_j = jh_\theta \ , \ j = 0, 1, \ldots, L \ , \ h_\theta = \frac{2\pi}{L}$$

$z_\infty$ denotes the outer boundary approximating infinity.
Solution of Streamfunction

The streamfunction is expanded into a truncated Fourier series

\[ \psi(z, \theta, t) = \frac{1}{2} F_0(z, t) + \sum_{n=1}^{N} \left[ F_n(z, t) \cos(n\theta) + f_n(z, t) \sin(n\theta) \right] \]

The Fourier coefficients satisfy

\[ \frac{\partial^2 F_n}{\partial z^2} - n^2 \lambda^2 F_n = \lambda^2 s_n(z, t) , \quad n = 0, 1, \ldots \]

\[ \frac{\partial^2 f_n}{\partial z^2} - n^2 \lambda^2 f_n = \lambda^2 r_n(z, t) , \quad n = 1, \ldots \]

At a fixed time these equations are effectively ODEs and are integrated using marching algorithms.
Solution of Streamfunction

The functions $r_n(z, t)$ and $s_n(z, t)$ are given by

\[
  s_n(z, t) = \frac{1}{\pi} \int_0^{2\pi} M^2 \zeta \cos(n\theta) d\theta
\]

\[
  r_n(z, t) = \frac{1}{\pi} \int_0^{2\pi} M^2 \zeta \sin(n\theta) d\theta
\]

and satisfy the integral conditions

\[
  \int_0^\infty e^{-n\lambda z} s_n(z, t) dz = 0 , \ n = 0, 1, 2, \ldots
\]

\[
  \int_0^\infty e^{-n\lambda z} r_n(z, t) dz = 0 , \ n = 1, 2, \ldots
\]
Solution of Vorticity & Temperature

The transport equations for $\zeta, \phi$ can be cast in generic form

$$ t \frac{\partial \chi}{\partial t} = q(z, \theta, t) $$

This equation is solved using the Crank-Nicholson implicit procedure. The solution is advanced from time $t$ to time $t + \Delta t$ by integrating the above

$$ \chi_{\tau}|_{t}^{t+\Delta t} - \int_{t}^{t+\Delta t} \chi d\tau = \int_{t}^{t+\Delta t} q d\tau $$

Approximating the integrals using the trapezoidal rule yields

$$ \chi(z, \theta, t + \Delta t) = \chi(z, \theta, t) + \left( \frac{\Delta t}{2t + \Delta t} \right) [q(z, \theta, t + \Delta t) + q(z, \theta, t)] $$

The resulting algebraic system is then solved iteratively.
Determination of Surface Vorticity

The surface vorticity is determined by inverting the expressions for $r_n$ and $s_n$. This leads to the truncated Fourier series

$$\zeta(0, \theta, t) = \frac{1}{M_0^2} \left\{ \frac{1}{2} s_0(0, t) + \sum_{n=1}^{N} [r_n(0, t) \sin(n\theta) + s_n(0, t) \cos(n\theta)] \right\}$$

The quantities $s_n(0, t)$ and $r_n(0, t)$ are computed by enforcing the integral conditions. That is, off the cylinder surface $r_n$ and $s_n$ can be computed using the most recent guess for $\zeta$. Then, $s_n(0, t)$ and $r_n(0, t)$ are computed by numerically satisfying the integral constraints.
The following steps are performed ($p \equiv$ iteration counter):
1. solve for $\phi^{(p)}(z, \theta, t + \Delta t)$,
2. solve for $\zeta^{(p)}(z, \theta, t + \Delta t)$ for $z \neq 0$,
3. compute $r_n^{(p)}(z, t + \Delta t)$, $s_n^{(p)}(z, t + \Delta t)$ for $z \neq 0$,
4. calculate $r_n^{(p)}(0, t + \Delta t)$, $s_n^{(p)}(0, t + \Delta t)$ by enforcing the integral conditions and hence compute $\zeta^{(p)}(0, \theta, t + \Delta t)$,
5. solve for $f_n^{(p)}(z, t + \Delta t)$, $F_n^{(p)}(z, t + \Delta t)$ and thus obtain $\psi^{(p)}(z, \theta, t + \Delta t)$,
6. repeat above steps till convergence is reached and increment $p$ by 1.

Convergence is reached when the difference between two successive iterates of the surface vorticity is less than $\epsilon$. 
After performing numerous numerical experiments, the following computational parameters were chosen:

\[ N = 25, \; \epsilon = 10^{-6}, \; z_\infty = 10 \]

A typical grid size used was \( K \times L = 200 \times 120 \). Because of the impulsive start, small time steps of \( \Delta t = 10^{-3} \) were used initially. As time increased the time step was gradually increased to \( \Delta t = 0.05 \). Results were obtained using values

\[ r = 0.5, \; \eta = 45^\circ, \; Pr = 0.7 \text{ for } Gr = 10^2 \text{ and } Gr = 10^4 \]
Isotherm plot for $Gr = 10^2, \eta = \frac{\pi}{4},$ $Pr = 0.7, r = 0.5$ at $t = 2.5$ (conduction regime).
Isotherm plot for
Gr = 10^2, \eta = \frac{\pi}{4},
Pr = 0.7, \ r = 0.5 \ at
\ t = 100 \ (well
developed \ plume).
Isotherm plot for
\(Gr = 10^4, \eta = \frac{\pi}{4},\)
\(Pr = 0.7, r = 0.5\) at \(t = 15.\)
Isotherm plot for
\[ Gr = 10^4, \eta = \frac{\pi}{4}, \]
\[ Pr = 0.7, r = 0.5 \] at \[ t = 20. \]
Isotherm plot for
Gr = 10^4, η = π/4,
Pr = 0.7, r = 0.5 at
t = 25.
Surface vorticity distributions for $Gr = 10^4$, $\eta = \frac{\pi}{4}$, $Pr = 0.7$, $r = 0.5$. 
Surface temperature distributions for
\( Gr = 10^4, \eta = \frac{\pi}{4}, \)
\( Pr = 0.7, r = 0.5. \)
For large $Gr$ and small $t$ it is possible to expand the flow variables in the double series:

$$\chi = \chi_0 + \lambda \chi_1 + \lambda^2 \chi_2 + \cdots$$

where each $\chi_n$ ($n = 0, 1, 2, \cdots$) is further expanded:

$$\chi_n(z, \theta, t) = \chi_{n0}(z, \theta) + t \chi_{n1}(z, \theta) + \cdots$$

The leading-order solution for the temperature is:

$$\phi(z, \theta, t) \sim \frac{2\sqrt{t}}{\sqrt{\pi Pr \sqrt{Gr}}} \left( e^{-PrM_0^2 z^2} - \sqrt{\pi PrM_0} \text{erfc}(\sqrt{PrM_0} z) \right)$$
Comparison of time variation of average surface temperature for $Gr = 10^4$, $\eta = \frac{\pi}{4}$, $Pr = 0.7$, $r = 0.5$. Good agreement for small $t$; agreement worsens with time.
Concluding Remarks

- Impulsively generated convection from an elliptic cylinder was investigated.
- The numerical method presented is successful for computing unsteady flows for a wide range of Grashof numbers.
- Numerical results were supported by analytical results.
- The technique can be easily extended to handle other cross sections.
- Future work includes comparisons with experiments ([4]).