Bénard Convection with Rotation and a Periodic Temperature Distribution

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Problem description

\[ \delta = \frac{H}{R} \ll 1 \]

\[ T = T_0 - \Delta T \]

\[ T = T_0 - \Delta T \cos(2\pi \theta) \]

\[ \text{rigid lid} \quad T = T_0 - \Delta T \quad z = H \]

\[ T = T_0 - \Delta T \cos\left(\frac{2\pi y}{\lambda}\right) \quad z = 0 \]
Previous work

- Pascal and D’Alessio [1] studied the stability of a flow with rotation and a quadratic density variation.
- Schmitz and Zimmerman [2] studied the effects of a spatially varying temperature boundary condition as well as wavy boundaries, without rotation and assuming a very large Prandtl number.
- Basak et al. [4], studied the flow resulting from a spatially varying temperature boundary condition in a square cavity without rotation.

The current work takes advantage of the thinness of the fluid layer, and investigates flow in a rectangular domain with rotation and a varying bottom temperature.
Governing equations and boundary conditions

Governing equations:

\[ \nabla \cdot \vec{v} = 0 \]
\[ \frac{\partial \vec{v}}{\partial t} + \left( \vec{v} \cdot \nabla \right) \vec{v} + f\hat{k} \times \vec{v} = \frac{-1}{\rho_0} \nabla p - \frac{\rho}{\rho_0} g\hat{k} + \nu \nabla^2 \vec{v} \]
\[ \frac{\partial T}{\partial t} + \left( \vec{v} \cdot \nabla \right) T = \kappa \nabla^2 T \]

Boundary conditions:

\[ u = v = w = 0 \text{ at } z = 0, H \text{ and } y = 0, \lambda \]
\[ T = T_0 - \Delta T \text{ at } z = H \text{ and } y = 0, \lambda \]
\[ T = T_0 - \Delta T \cos \left( \frac{2\pi y}{\lambda} \right) \text{ at } z = 0 \]
Other conditions

Initial conditions:

\[ u = v = w = 0 \ , \ T = T_0 - \frac{\Delta T}{H}z - \Delta T \cos \left( 2\pi \frac{y}{\lambda} \right) \left( 1 - \frac{Z}{H} \right) \] at \ t = 0

Density is assumed to vary according to:

\[ \rho = (1 - \alpha [T - T_0]) \]

The flow is assumed to be uniform in the \( x \)-direction. This allows a stream function and vorticity to be defined as:

\[ \mathbf{v} = \frac{\partial \psi}{\partial z} \ , \ \mathbf{w} = -\frac{\partial \psi}{\partial y} \ , \ \zeta = - \left( \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \right) \]
Other conditions

Integral constraints are imposed on the vorticity and can be derived using Green’s Second Identity:

\[
\int_V \left( \phi \nabla^2 \chi - \chi \nabla^2 \phi \right) dV = \int_S \left( \phi \frac{\partial \chi}{\partial n} - \chi \frac{\partial \phi}{\partial n} \right) dS
\]

Here \( \phi \) and \( \chi \) denote arbitrary differentiable functions, \( \frac{\partial}{\partial n} \) is the normal derivative, and \( S \) is the surface enclosing the volume \( V \). Choosing \( \phi \) to satisfy \( \nabla^2 \phi = 0 \) and letting \( \chi \equiv \psi \), then \( \nabla^2 \chi = \nabla^2 \psi = -\zeta \). Applying the boundary conditions \( \frac{\partial \psi}{\partial n} = \psi = 0 \) on \( S \), the above leads to

\[
\int_0^H \int_0^\lambda \phi_n \zeta dydz = 0
\]

where

\[
\phi_n(y, z) = e^{\pm 2n\pi z} \left\{ \begin{array}{ll}
\sin(2n\pi y) & \text{for } n = 1, 2, 3, \cdots \\
\cos(2n\pi y) &
\end{array} \right.
\]
Scaling and dimensionless parameters

\[ t \to \frac{H^2}{\kappa} t, \ y \to \lambda y, \ z \to Hz, \ \psi \to \kappa \psi, \ \zeta \to \frac{\kappa}{H^2} \zeta \]

\[ T \to (T_0 - \Delta T) + \Delta TT, \ u \to \frac{\kappa}{H} u \]

\[ Ra = \frac{\alpha g H^3 \Delta T}{\nu \kappa} \] Rayleigh number

\[ Ro = \frac{\kappa}{Hf \lambda} \] Rossby number

\[ Pr = \frac{\nu}{\kappa} \] Prandtl number

\[ \delta = \frac{H}{\lambda} \] Aspect ratio
Dimensionless equations

\[
\frac{\partial \zeta}{\partial t} - \delta \frac{\partial}{\partial z} \left( \frac{\partial \psi}{\partial z} \frac{\partial^2 \psi}{\partial y \partial z} - \frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial z^2} \right) + \delta^3 \frac{\partial}{\partial y} \left( - \frac{\partial \psi}{\partial z} \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial y \partial z} \right) - \frac{\delta}{Ro} \frac{\partial u}{\partial z} \\
= \delta PrRa \frac{\partial T}{\partial y} + Pr \left( \delta^2 \frac{\partial^2 \zeta}{\partial y^2} + \frac{\partial^2 \zeta}{\partial z^2} \right)\\
\zeta = - \left( \delta^2 \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \right)\\
\frac{\partial u}{\partial t} + \delta \left( \frac{\partial \psi}{\partial z} \frac{\partial u}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial u}{\partial z} \right) - \frac{\delta}{Ro} \frac{\partial \psi}{\partial z} = Pr \left( \delta^2 \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)\\
\frac{\partial T}{\partial t} + \delta \left( \frac{\partial \psi}{\partial z} \frac{\partial T}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial T}{\partial z} \right) = \delta^2 \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}
\]
Approximate analytical solution

For small $\delta$, an approximate analytical solution can be constructed:

\[ \psi = \psi_0 + \delta \psi_1 + \cdots, \quad \zeta = \zeta_0 + \delta \zeta_1 + \cdots \]

\[ u = u_0 + \delta u_1 + \cdots, \quad T = T_0 + \delta T_1 + \cdots \]

The leading-order problem becomes:

\[ \frac{\partial T_0}{\partial t} = \frac{\partial^2 T_0}{\partial z^2}, \quad \frac{\partial \zeta_0}{\partial t} = Pr \frac{\partial^2 \zeta_0}{\partial z^2} + Pr \delta Ra \frac{\partial T_0}{\partial y} \]

\[ \frac{\partial^2 \psi_0}{\partial z^2} = -\zeta_0, \quad \frac{\partial u_0}{\partial t} = Pr \frac{\partial^2 u_0}{\partial z^2} \]
Approximate analytical solution

Applying the boundary, initial and integral conditions yields:

\[
\zeta_0(y, z, t) = -\pi \delta Ra \left( z^2 - \frac{z^3}{3} - \frac{7z}{10} + \frac{1}{10} \right) \sin(2\pi y)
\]

\[
+ \sum_{n=1}^{\infty} a_n e^{-n^2 \pi^2 Prt} \cos(n\pi z)
\]

\[
\psi_0(y, z, t) = \pi \delta Ra \left( \frac{z^4}{12} - \frac{z^5}{60} - \frac{7z^3}{60} + \frac{z^2}{20} \right) \sin(2\pi y)
\]

\[
- \sum_{n=1}^{\infty} \frac{a_n}{n^2 \pi^2} e^{-n^2 \pi^2 Prt} (1 - \cos(n\pi z))
\]

where

\[
a_n = 2\pi \delta Ra \sin(2\pi y) \int_{0}^{1} \left( z^2 - \frac{z^3}{3} - \frac{7z}{10} + \frac{1}{10} \right) \cos(n\pi z) dz , \ n = 1, 2, \ldots
\]
Steady-state solutions

As \( t \to \infty \), the following steady-state solutions emerge:

\[
T_s = (1 - z)(1 - \cos(2\pi y))
\]

\[
\zeta_s = -\pi \delta Ra \left( z^2 - \frac{z^3}{3} - \frac{7z}{10} + \frac{1}{10} \right) \sin(2\pi y)
\]

\[
\psi_s = \pi \delta Ra \left( \frac{z^4}{12} - \frac{z^5}{60} - \frac{7z^3}{60} + \frac{z^2}{20} \right) \sin(2\pi y)
\]

\[u_s = 0\]

Plotted on the next slide are the leading-order temperature and velocities \((Ro = 0.0548, Pr = 0.7046 \text{ and } Ra = 388.7)\).
Steady-state solutions
At leading order $u_s = 0$, the $O(\delta)$ solution for $u_s$ is:

$$u_s = \frac{1}{720} \frac{\pi \delta Ra}{RoPr} \sin(2\pi y)z(z - 1) \left(2z^4 - 10z^3 + 11z^2 - z - 1\right)$$
Transient development

$\nu$- velocity development with time for $Ra = 1$, $Pr = 1$ at times $t = 0.01, 0.1, 1$ from top to bottom.
Transient development

\( w \)- velocity development with time for \( Ra = 1, \ Pr = 1 \) at times \( t = 0.01, 0.1, 1 \) from top to bottom.
The steady-state solution was also be determined numerically using the commercial software package CFX. All results shown use air at 298K with a domain having a length of 20 cm and height of 2 cm with periodicity imposed at the ends. For cases with rotation, the angular velocity is $0.05s^{-1}$. The non-dimensional values are $Ro = 0.0548$ and $Pr = 0.7046$. The Rayleigh number will depend on $\Delta T$. 
Steady-state numerical solutions

The temperature distribution for a case without rotation or modulated bottom heating (top) is compared to a case with these effects ($\Delta T = 0.5K$, $Ra = 388.7$).
Steady-state numerical solutions

The velocity for a case without rotation or modulated bottom heating (top) is compared to a case with these effects ($\Delta T = 0.5K$, $Ra = 388.7$).

Note that the pattern and magnitude of the temperature and velocity agree well with the approximate analytical solutions.
Unstable numerical solutions

$\Delta T = 0.5 K, \ Ra = 388.7$

$\Delta T = 4 K, \ Ra = 3109$
Unstable solutions

Unstable numerical solutions

- \( dT = 1.5 \text{ K} \)
- \( dT = 1.8 \text{ K} \)
- \( dT = 2.0 \text{ K} \)
- \( dT = 2.5 \text{ K} \)
- \( dT = 6.0 \text{ K} \)
- \( dT = 12 \text{ K} \)
Unstable numerical solutions

- dT = 1.5 K
  - Temperature distribution
  - Speed range: 0 m/s to 3.657e-3 m/s

- dT = 1.8 K
  - Temperature distribution
  - Speed range: 0 m/s to 6.058e-3 m/s

- dT = 2.0 K
  - Temperature distribution
  - Speed range: 0 m/s to 9.089e-3 m/s

- dT = 2.5 K
  - Temperature distribution
  - Speed range: 0 m/s to 1.343e-2 m/s

- dT = 6.0 K
  - Temperature distribution
  - Speed range: 0 m/s to 3.072e-2 m/s

- dT = 12 K
  - Temperature distribution
  - Speed range: 0 m/s to 4.583e-2 m/s
Summary

Conclusions

- Contrary to the Bénard problem, a non-zero background flow was found.
- The approximate analytical solution agrees well with the fully numerical solution.
- While rotation is known to stabilize the flow, a variable bottom temperature also influences the stability of the flow.
- Interesting features emerging from the unstable numerical simulations were observed.

