A Mathematical and Numerical Study of Roll Waves

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Unstable flow down an incline

- Critical conditions for the onset of Instability.
- Structure of Roll Waves
- Investigate the effect of bottom topography
The spillway from the Llyn Brianne Dam in Wales
Figure 2.1: The picture on the left shows a laboratory experiment in which roll waves appear on water flowing down an inclined channel. The fluid is about 7 mm deep and the channel is 10 cm wide and 18 m long; the flow speed is roughly 65 cm/sec. Time series of the free-surface displacements at four locations are plotted in the pictures on the right. In the upper, right-hand panel, small random perturbations at the inlet seed the growth of roll waves whose profiles develop downstream (the observing stations are 3 m, 6 m, 9 m and 12 m from the inlet and the signals are not contemporaneous). The lower right-hand picture shows a similar plot for an experiment in which a periodic train was generated by moving a paddle at the inlet; as that wavetrain develops downstream, the wave profiles become less periodic and there is a suggestion of subharmonic instability.

Experiment taken from Balmforth & Mandre (JFM, 2004)
Coordinate system

\[ \zeta(x) \]

\[ x, u \]

\[ z, w \]

\[ h(x, t) \]

\[ \theta \]
Equations of motion

\[ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \]

\[ \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + g \rho \sin \theta + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} \right) \]

\[ \frac{1}{\rho} \frac{\partial p}{\partial z} + g \cos \theta - \frac{\mu}{\rho} \frac{\partial^2 w}{\partial z^2} = 0 \]

Assumed \( Re \sim O(1) \) and neglected terms \( O(\delta^2) \) and higher

where \( \delta = H/L \) is the aspect ratio
Interface conditions

Free surface conditions:

\[
\begin{align*}
\rho - 2\mu \frac{\partial w}{\partial z} &= 0 \\
\mu \frac{\partial u}{\partial z} &= 0 \\
w &= \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + u\zeta'(x)
\end{align*}
\]

at \( z = \zeta(x) + h(x, t) \)

Bottom boundary conditions:

\[
\begin{align*}
u + \zeta'(x)w &= 0 \quad \text{and} \quad \zeta'(x)u - w &= 0 \quad \text{at} \quad z = \zeta(x)
\end{align*}
\]

\[\Rightarrow u = w = 0 \quad \text{at} \quad z = \zeta(x)\]
Integral boundary layer (IBL) method

Depth-integrate equations and introduce flow variables

\[ h(x, t) \text{ and } q(x, t) = \int_{\zeta}^{\zeta+h} u \, dz \]

To convert terms

\[ \int_{\zeta}^{\zeta+h} u^2 \, dz \text{, } \frac{\mu}{\rho} \frac{\partial u}{\partial z} \bigg|_{z=\zeta} \]

assume the parabolic velocity profile:

\[ u(x, z, t) = \frac{3q}{2h^3} \left[ 2(h + \zeta)z - z^2 - (\zeta + 2h)\zeta \right] \]
Dimensionless equations

In terms of \( h, q \) the dimensionless equations become

\[
\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0
\]

\[
\frac{\partial q}{\partial t} + \frac{6}{5} \frac{\partial}{\partial x} \left( \frac{q^2}{h} \right) = \frac{1}{Fr^2} \left( h - h \frac{\partial h}{\partial x} - \zeta'(x)h - \frac{q}{h^2} \right)
\]

\[
+ \frac{3Fr^2}{Re^2} \left[ \frac{7}{2} \frac{\partial^2 q}{\partial x^2} - \frac{9}{h} \frac{\partial q}{\partial x} \frac{\partial h}{\partial x} + \frac{9q}{h^2} \left( \frac{\partial h}{\partial x} \right)^2 - \frac{9q}{2h} \frac{\partial^2 h}{\partial x^2} 
\right]
\]

\[
- \frac{6\zeta'(x)}{h} \frac{\partial q}{\partial x} + \frac{6\zeta'(x)q}{h^2} \frac{\partial h}{\partial x} - \frac{3\zeta''(x)q}{h} - \frac{6(\zeta'(x))^2 q}{h^2}
\]

where \( Fr^2 = \frac{Re}{3 \cot \theta} \), \( Re = \frac{\rho Q}{\mu} \) and \( \zeta(x) = a_b \cos(k_b x) \)
Linear stability: \( a_b = 0 \) case

The steady-state flow is: \( q_s = h_s = 1 \)

Imposing disturbances on this steady flow and linearizing yields the dispersion equation

\[
Fr^2 \sigma^2 + \left( \frac{21 Fr^4}{2 Re^2} k^2 + 1 + i \frac{12}{5} Fr^2 k \right) \sigma + \left( 1 - \frac{6}{5} Fr^2 \right) k^2 \\
+ i \left( 3k + \frac{27 Fr^4}{2 Re^2} k^3 \right) = 0
\]

where \( \sigma \) is the growth rate and \( k \) is the wavenumber of the disturbance
Linear stability results for $a_b = 0$

The flow is stable if $Fr < \frac{1}{\sqrt{3}}$ while for $Fr > \frac{1}{\sqrt{3}}$ instability occurs for wavenumbers $k < k_{max}$ where

$$k_{max} = \frac{10Re}{\sqrt{30Fr^2}} \sqrt{\frac{3Fr^2 - 1}{3Fr^2 + 35 + 12Fr\sqrt{6Fr^2 + 25}}}$$

For large $Fr$ the asymptotic behaviour is

$$k_{max} \sim \frac{10Re}{Fr^2 \sqrt{30(1 + 4\sqrt{6})}}$$
Neutral stability curves have a maximum at $Fr \approx 0.76286$ (independent of $Re$).
Linear stability: \( a_b \neq 0 \) case

The steady state solution is \( q_s = 1 \) and \( h_s(x) \) satisfies

\[
3\beta [h_s h''_s - 2(h'_s)^2] + (2\alpha h^3_s - 4\beta \zeta' - \frac{12}{5}) h'_s \\
+ 2\beta \zeta'' h_s - 2\alpha (1 - \zeta') h^3_s = -2\alpha - 4\beta (\zeta')^2
\]

where \( \alpha = \frac{1}{Fr^2} \) and \( \beta = 9 \left( \frac{Fr}{Re} \right)^2 \)

An approximate solution can be constructed in the form

\[
h_s(x) = 1 + (a_b k_b) h_s^{(1)}(x) + (a_b k_b)^2 h_s^{(2)}(x) + \cdots
\]
Periodic steady state solution

\[ Fr = 1, \ Re = 10, \ a_b = 0.1, \ k_b = 2\pi \]
Linear stability: \( ab \neq 0 \) case

To study how small disturbances will evolve, introduce perturbations \( \hat{h}, \hat{q} \) superimposed on the steady-state solution and linearize equations using

\[
h = h_s(x) + \hat{h}, \quad q = 1 + \hat{q}
\]

For an uneven bottom, the coefficients in the linearized equations are periodic functions; hence apply Floquet-Bloch theory to conduct the stability analysis and represent the perturbations as Bloch-type functions having the form

\[
\hat{h} = e^{\sigma t} e^{iKx} \sum_{n=-\infty}^{\infty} \hat{h}_n e^{ink_b x}, \quad \hat{q} = e^{\sigma t} e^{iKx} \sum_{n=-\infty}^{\infty} \hat{q}_n e^{ink_b x}
\]
Numerical linear stability results for \( a_b \neq 0 \)

Critical Froude number as a function of bottom amplitude with \( Re = 10 \)
Numerical linear stability results for $a_b \neq 0$

Critical Froude number as a function of bottom amplitude with $k_b = 2\pi$
Numerical linear stability results for $a_b \neq 0$

Neutral stability curves for the case with $Re = 10$ and $k_b = 2\pi$
Numerical linear stability results for $a_b \neq 0$

Neutral stability curves for the case with $Re = 10$ and $a_b = 0.2$
Begin by expressing equations in the form

\[
\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0
\]

\[
\frac{\partial q}{\partial t} + \frac{\partial}{\partial x} \left( \frac{6}{5} q^2 \frac{1}{h} + \frac{\alpha}{2} h^2 \right) = \psi(h, q) + \chi \left( x, h, q, \frac{\partial h}{\partial x}, \frac{\partial q}{\partial x}, \frac{\partial^2 h}{\partial x^2}, \frac{\partial^2 q}{\partial x^2} \right)
\]

where \( \psi = \alpha \left( h - \frac{q}{h^2} \right) \)

and \( \chi = -\alpha \zeta' h - 2\beta \zeta' \left( \zeta' - \frac{\partial h}{\partial x} \right) \frac{q}{h^2} - \beta \zeta'' \frac{q}{h} - 2\beta \frac{\zeta'}{h} \frac{\partial q}{\partial x} \)

\[+\beta \left( \frac{7}{6} \frac{\partial^2 q}{\partial x^2} - \frac{3}{2} \frac{q \partial^2 h}{h \partial x^2} - \frac{3}{h} \frac{\partial q}{\partial x} \frac{\partial h}{\partial x} + 3 \frac{q}{h^2} \left( \frac{\partial h}{\partial x} \right)^2 \right) \]
Fractional-step method (LeVeque, 2002)

Decouple the advective and diffusive components, first solve

\[
\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0
\]

\[
\frac{\partial q}{\partial t} + \frac{\partial}{\partial x} \left( \frac{6}{5} \frac{q^2}{h} + \frac{\alpha}{2} h^2 \right) = \psi(h, q)
\]

over a time step \( \Delta t \), and then solve

\[
\frac{\partial q}{\partial t} = \chi \left( x, h, q, \frac{\partial h}{\partial x}, \frac{\partial q}{\partial x}, \frac{\partial^2 h}{\partial x^2}, \frac{\partial^2 q}{\partial x^2} \right)
\]

using the solution obtained from the first step as an initial condition for the second step; the second step returns the solution for \( q \) at the new time \( t + \Delta t \)
First step

This involves solving a nonlinear system of hyperbolic conservation laws; express in vector form

\[
\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = b(U)
\]

where \( U = \begin{bmatrix} h \\ q \end{bmatrix} \), \( F(U) = \begin{bmatrix} \frac{6}{5} \frac{q^2}{h} + \frac{\alpha}{2} h^2 \\ q \end{bmatrix} \), \( b(U) = \begin{bmatrix} 0 \\ \psi \end{bmatrix} \)

Utilize MacCormack’s method to solve this system; this is a conservative second-order accurate finite difference scheme which correctly captures discontinuities and converges to the physical weak solution of the problem.
LeVeque & Yee (JCP, 1990) extended MacCormack’s method to include source terms; this explicit predictor-corrector scheme takes the form

\[ U_j^* = U_j^n - \frac{\Delta t}{\Delta x} \left[ F(U_{j+1}^n) - F(U_j^n) \right] + \Delta t \, b(U_j^n) \]

\[ U_{j+1}^n = \frac{1}{2} \left( U_j^n + U_j^* \right) - \frac{\Delta t}{2\Delta x} \left[ F(U_j^*) - F(U_{j-1}^*) \right] + \frac{\Delta t}{2} b(U_j^*) \]

where the notation \( U_j^n \equiv U(x_j, t_n) \) was adopted, \( \Delta x \) is the grid spacing and \( \Delta t \) is the time step; second-order accuracy is achieved by first forward differencing and then backward differencing.
Second step

This reduces to solving the generalized one-dimensional linear diffusion equation given by:

\[
\frac{\partial q}{\partial t} = \frac{7 \beta}{6} \frac{\partial^2 q}{\partial x^2} + S_1 \frac{\partial q}{\partial x} + S_0 q + S
\]

where \( S = -\alpha \zeta' h \) and

\[
S_0 = -\beta \frac{\zeta''}{h} - 2 \beta \frac{\zeta'}{h^2} \left( \zeta' - \frac{\partial h}{\partial x} \right) - \frac{3}{2} \beta \frac{\partial h}{h} \frac{\partial^2 h}{\partial x^2} + 3 \beta \frac{h}{h^2} \left( \frac{\partial h}{\partial x} \right)^2
\]

and \( S_1 = -2 \beta \frac{\zeta'}{h} - 3 \beta \frac{\partial h}{h} \frac{\partial x}{\partial x} \)
The problem is completely specified by $Fr$, $Re$, $a_b$ and $k_b$; typical computational parameters used were:

Computational Domain: $0 \leq x \leq L$

with $\lambda_b \leq L \leq 300\lambda_b$, $\lambda_b = \frac{2\pi}{k_b}$

Grid Spacing: $\Delta x = 0.01$

Time Step: $\Delta t = 0.002$
Parameters:
\( a_b = 0.1, k_b = 2\pi, \)
\( Re = 10, Fr = 0.7 \)
A subharmonic instability known as wave coarsening occurs for \( L = 45\lambda_b \)
Evolution of flow rate

Parameters:
\( a_b = 0.1, \quad k_b = 2\pi, \quad Re = 10, \quad Fr = 0.7 \)

Interruption in wave coarsening occurs for \( L = 30\lambda_b \)
Wave spawning

Parameters:
\( a_b = 0.1, k_b = 2\pi, \)
\( Re = 10, Fr = 0.7 \)

A wave-spawning instability occurs for
\( L = 300\lambda_b \)
A mathematical model of roll waves along with a numerical method to solve the model were presented.

Investigated the effect of sinusoidal bottom topography on the formation of roll waves.

Bottom topography has a stabilizing effect on the flow for small to moderate waviness parameters.

Future work includes repeating the analysis for a porous wavy bottom and to include surface tension.