Phase Models and oscillators with Delayed, All-to-all Coupling

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Problem

Time delays are common in the connections between oscillators due to the time for signal propagation. We investigate the effects of delays in the coupling behavior.

Phase model analysis

Consider a network of identical oscillators with time delayed, and all-to-all coupling

$$\frac{dX_i}{dt} = F(X_i(t)) + \epsilon \sum_{j=1, j\neq i}^{N} G(X_i(t), X_j(t-\tau)), \quad (1)$$

where $i = 1, \ldots, N$. When ϵ is sufficiently small and the delay is sufficiently small ($\Omega\tau = O(1)$ with respect to ϵ), the appropriate phase model is

$$\frac{d\phi_i}{dt} = \Omega + \epsilon \sum_{j=1, j\neq i}^{N} H(\phi_j - \phi_i - \eta), \quad (2)$$

with $i = 1, \ldots, N$.

Symmetric cluster states: each cluster contains the same number of oscillators.

n: number of clusters. Φ_k, $k = 0, \ldots, n-1$, the phase of cluster k. The symmetry of the system implies that if one such a solution exists, then there is a whole family. Without loss of generality, we assume the oscillators cluster in order of their indices. That is, $\phi_i = \Phi_k$, $i = k \cdot \frac{N}{n} + 1, \ldots, (k+1) \cdot \frac{N}{n}$, $k = 0, \ldots, n-1$.

Assume that

$$\Phi_k = (\tilde{\Omega} + \omega^{(n)})t + \frac{2\pi k}{n}, \quad (3)$$

which means that the n clusters are equally separated in phase.

Existence and stability

Existence: The cluster solution in the form (3) exists if

$$\omega^{(n)} = \epsilon \frac{N}{n} \sum_{m=0}^{n-1} H(\frac{2\pi m}{n} - \eta).$$

Stability: By linearization, we have the $N-1$ eigenvalues for n-cluster solution:

$$\lambda_0^{(n)} = -\frac{N}{n} \sum_{k=0}^{n-1} H'(\frac{2\pi k}{n} - \eta), \text{ multiplicity } N-n.$$

$$\lambda_p^{(n)} = -\frac{N}{n} \sum_{k=0}^{n-1} H'(\frac{2\pi k}{n} - \eta)(1 - e^{i2\pi kp/n}),$$

with $p = 1, \cdots, n-1$.

Stability remarks

Conclusions about stability of symmetric cluster solutions:

1. If $n < N$, the stability of n-cluster solutions depends on number of clusters and phase differences, not on the size of the network.
2. The 1-cluster (synchronization) solution always exists, and is asymptotically stable if $H'(-\eta) > 0$.
3. If N is even, 2-cluster solutions always exist and are asymptotically stable if $H'(\pi - \eta) > 0$ and $H'(-\eta) + H'(\pi - \eta) > 0$.
4. We assumed $\epsilon > 0$. If $\epsilon < 0$, the stability of asymptotically stable solutions and totally unstable solutions will be reversed, while saddle type solutions remain of saddle type.

Application to a network of Morris-Lecar oscillators

The model:

$$v_i' = I_{app} - g_{Ca} m_\infty(v_i)(v_i - v_{Ca}) - g_K w_i (v_i - v_K) - g_L(v_i - v_L) - \frac{g_{syn}}{N-1} \sum_{j=1, j\neq i}^{N} s(v_j(t-\tau)) v_i(t),$$

$$w_i' = \varphi \lambda(v_i)(w_\infty(v_i) - w_i),$$

where $i = 1, \ldots, N$ and

$$m_\infty(v) = \frac{1}{2}(1 + \tanh((v - \nu_1)/\nu_2)), \quad \lambda(v) = \cosh((v - \nu_3)/2\nu_4),$$

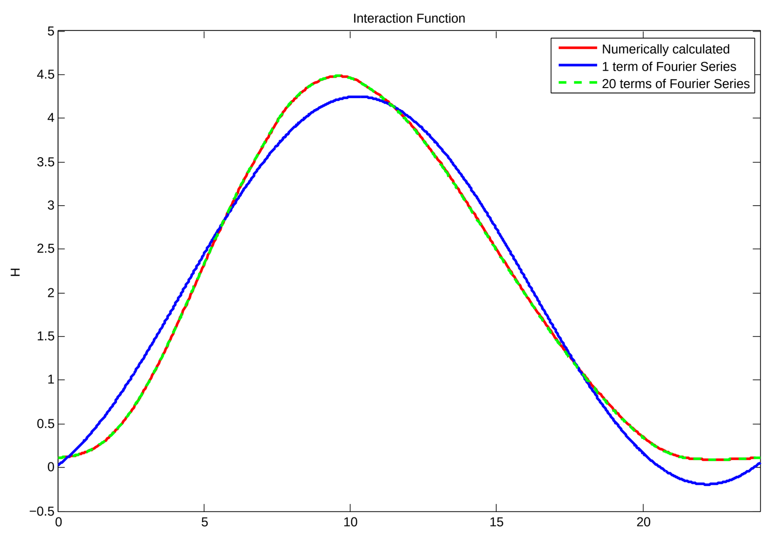
$$w_\infty(v) = \frac{1}{2}(1 + \tanh((v - \nu_3)/\nu_4)), \quad s(v) = \frac{1}{2}(1 + \tanh(10v)).$$

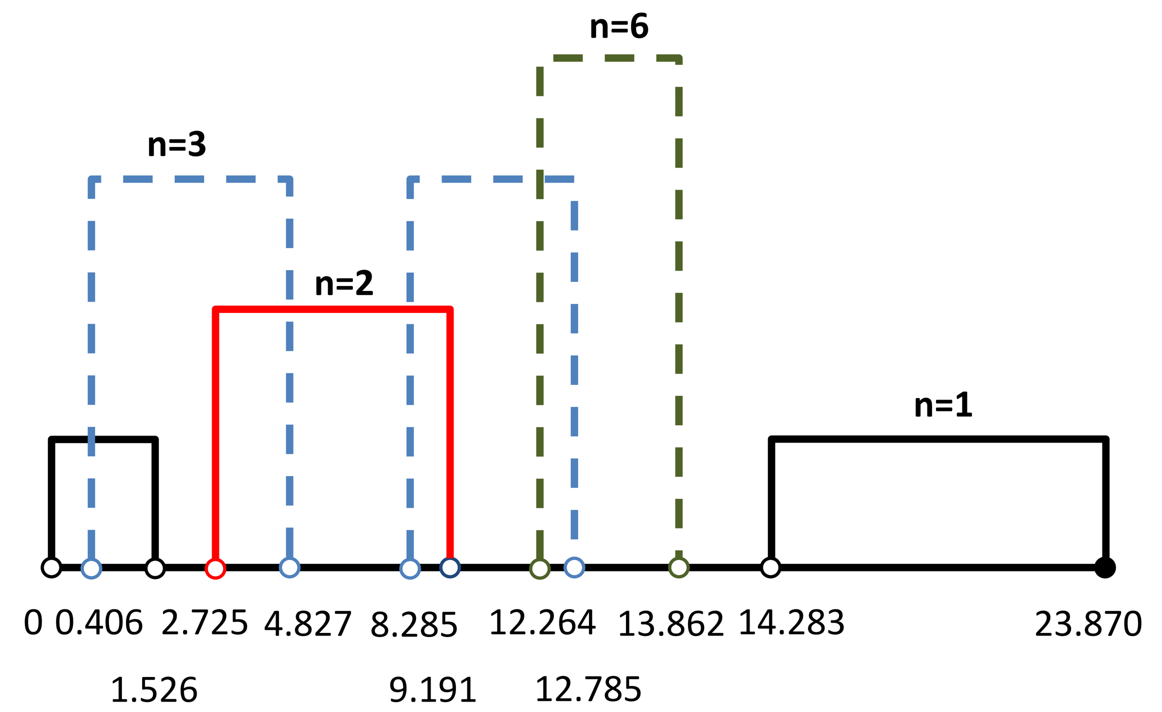
Using parameter set I in Campbell and Kobelevskiy (2012), each neuron with no coupling ($\epsilon = 0$) has a unique exponentially asymptotically stable limit cycle with period $T \approx 23.87$ corresponding to $\Omega = 0.2632$.

Phase model analysis The corresponding phase model is

$$\frac{d\phi_i}{dt} = \Omega - \epsilon \sum_{j=1, j\neq i}^{N} H(\phi_j - \phi_i - \eta), \; i = 1, \ldots, N.$$

where $H(\phi) = a_0 + \sum_{k=1}^{K}(a_k \cos(k\phi) + b_k \sin(k\phi))$.





Numerical study:

n	$g_{syn} = 0.01$	$g_{syn} = 0.1$
1	$[0, 1.6) \cup (13.4, 23.9]$	$[0, 1.7) \cup (9.4, 23.9]$
2	$(2.4, 9.1)$	$(1.9, 8.0) \cup (20.2, 23.9]$
3	$(0.6, 4.9) \cup (8.5, 12.8)$	$(0.1, 3.9) \cup (7.2, 11.2) \cup (20.2, 23.3)$
6	$(12.4, 13.7)$	$(0.2, 1.0) \cup (11.0, 12.0)$

References

[1] S. A. Campbell and I. Kobelevskiy. Phase modles and oscillators with time delayed coupling, In *Dis. Cont. Dyn. Sys.* '12

[2] K. Okuda. Variety and generality of clustering in globally coupled oscillators, In *Physica D* '93

A Future Direction

We focussed on all-to-all coupling. However, other types of connectivity with symmetry can be analyzed in similar manner.

Our work is restricted to phase model solutions, symmetric cluster solutions and weak coupling. Relaxing these restrictions will require other approaches, such as equivariant bifurcation theory.

Acknowledgements

This work has benefitted from the support of the Natural Sciences and Engineering Research Council of Canada and the Faculty of Mathematics of University of Waterloo.