· Trisections are a decomposition of 4-manifolds that take inspiration from Heegaard splittings - talk about this first.

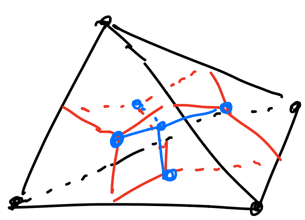
Defn: A Heegaard splitting of M^3 is a decomposition $M = H_g \cup_{\Sigma_g} H_g$, where $H_g = \natural^g S^1 \times B^2$, $\partial H_g = \Sigma_g$.

(Recall: $\partial(X \natural Y) = \partial X \# \partial Y$)

there's a proof of existence:

→ 3-manifolds are triangulable [Möise, Bing 1952] but this is a hard fact that depends on $\dim M = 3$.

· A regular nbhd of the 1-skeleton is a handlebody: it's complement is <u>also</u> a handlebody since this is a regular nbhd of the dual simplex.



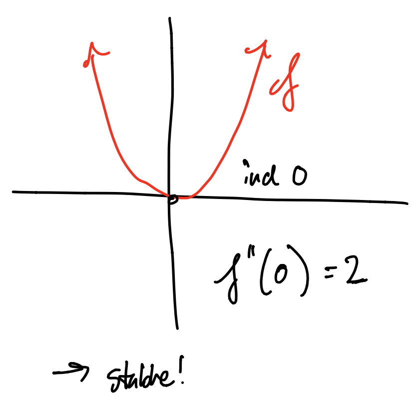
→ It's not entirely obvious this is smooth, but it's true. □

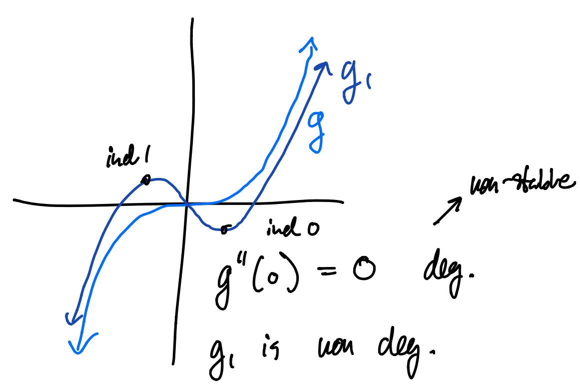
We will give another proof that is more adapted to higher dimensions. We first need some Morse theory.

<u>Defn</u>: A function $f: M \to \mathbb{R}$ is called Morse if it has no degenerate critical points.

(Recall: degenerate means $\text{Hess} f_p$ is singular)

eg.



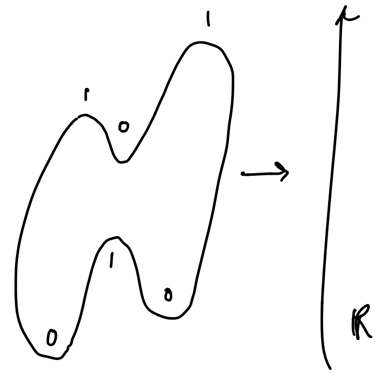


<u>Moral</u>: generic functions are Morse: they're dense (open) in the set of smooth functions.

The index of a critical point is $\text{index}(\text{Hess} f)_p$; ie the number of negative eigenvalues.

<u>Morse Lemma</u>: Let $f: M \to \mathbb{R}$, $p \in M$ be a non-deg. critical point: Then $\exists$ coordinate charts st. $f = -x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2$. (for index k)

Proof is bare hands, multivariable calc.

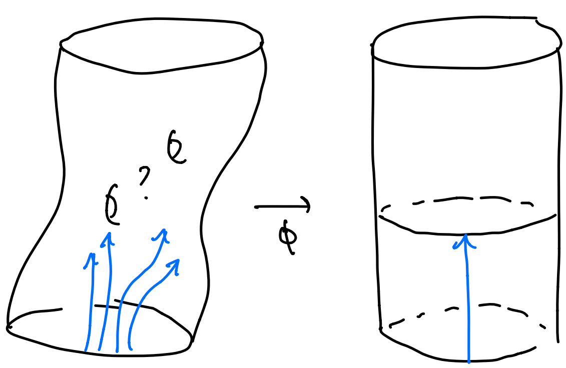


__Remarks:__

- critical pts are isolated.
- want to move/cancel critical pts.

__Theorem:__ Let X be a cobord from M_0 to M_1, and $f: X \to \mathbb{R}$ be Morse & have __no__ critical points. Then $X \cong M_0 \times I$; in particular $M_0 \cong M_1$.

__Proof:__ Pick g. Then $\phi: X \to M_0 \times I$ is defined by flow along $\nabla_g f$. This is a diffeo by ODE theorem.



Moreover, f transforms to proj_2 under ϕ.

no critical pts.

Take $W = (df)^\#$; then set $V = \frac{W}{\|W\|}$. Then $\|V\| = 1$.

$$df(V) \equiv g(W, V) \equiv \frac{1}{\|W\|} g(W, W) \equiv 1.$$

$\phi(t,p) =$ "flow along V by time t from p".

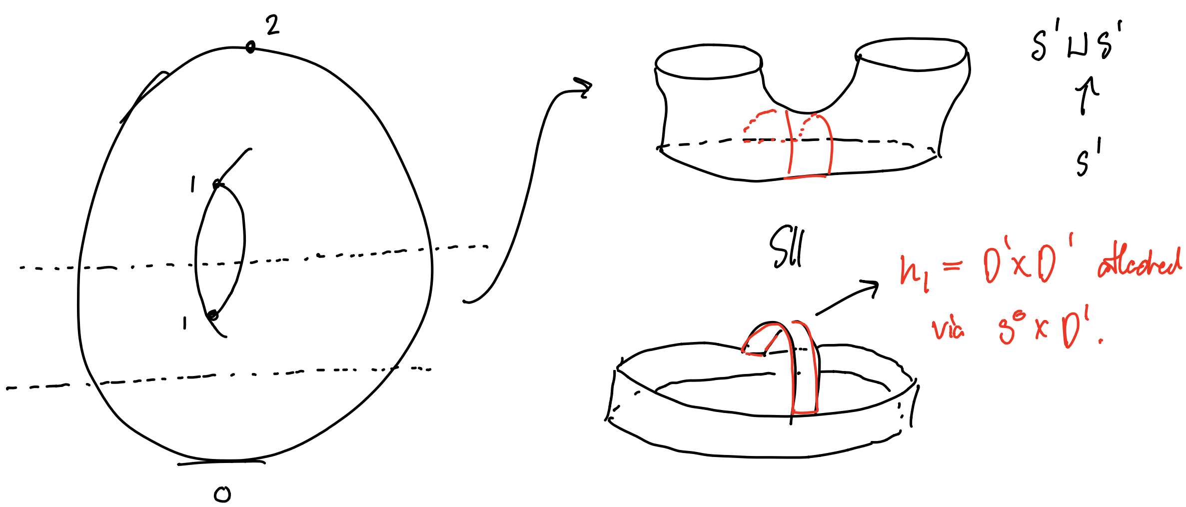
Details to check, but this works. $\square$

→ Moral: If we don't see critical pts, topology is a product. When there is a critical pt: topology changes.

<u>Theorem</u>: Let $f : X \to \mathbb{R}$ have a single critical point of index k. Then $X \cong (M_0 \times I) \cup \{k\text{-handle}\}$.

<u>Proof</u>: Picture. $h_k = D^k \times D^{n-k}$, attached along $S^k \times D^{n-k}$ comp.

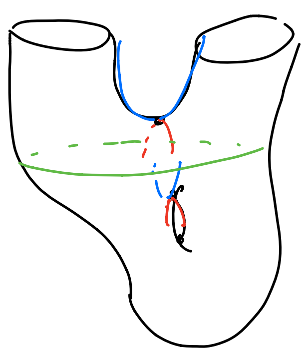
<u>eg.</u> 1-handle: $D^1 \times D^{n-1}$ attached along $S^0 \times D^{n-1}$. etc.



$\square$

Moral: Morse functions give you a CW-decomp.

Theorem: Let $f: X \to \mathbb{R}$ Morse with critical points of index k, ℓ st. $k < \ell$. Then we can slide the index k pts beneath the index ℓ one.

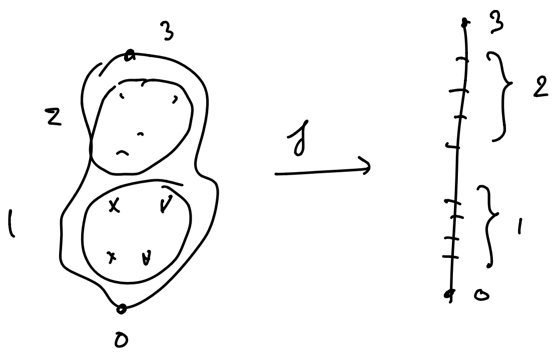


ascending/descending manifolds intersect transversely in $f^{-1}(\ell)$.

With the right dimension count, this means not at all

Moral: We can arrange the critical points for f to be in increasing order.

Back to Heegaard splittings:



(Same # of 1's as 2's)

$f^{-1}(\frac{3}{2}) = \Sigma_g$; Moreover, $f^{-1}([0,\frac{3}{2}]) \cong H_g \cong f^{-1}([\frac{3}{2},1])$

Which is the desired decomposition! Great.

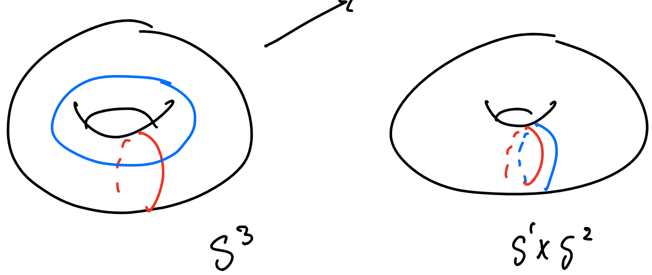
→ Heegaard splittings can equiv. be thought of as taking $\Sigma \times I$ and attaching 2-handles in both directions.
Ie, $D^2 \times D^1$ along $S^1 \times D^1$; thickened curves.

<u>Claim</u>: isotopy class of embeddings doesn't matter

<u>Claim</u>: Cap uniquely with B^3's. [Cerf's theorem]

With these: diagrams!

do this one!



<u>Theorem</u> [Reidemeister - Singer] Every 3-manifold admits a Heegaard splitting. Moreover any two given splittings are stably isotopic. (connect sum w/ S^3's)

Diagrammatically, this means any two diagrams are handleslide-diffeo equivalent eventually.

__Theorem__: [Waldhausen] Every splitting of S^3 is stabilized.

__Theorem__: [Haken's lemma] Heegaard splittings of reducible manifolds are reducible.

__Remarks__: Heegaard splittings / diagrams are useful for various things:

- Heegaard-Floer homology.
- minimal surfaces are HS?

Now: trisections. There's a magic theorem.

__Theorem__: [Laudenbach-Poénaru] Every diffeo. of $\# S^1 \times S^2$ extends across $\natural S^1 \times B^3$.

→ Note! This means we __can't__ bisect 4-manifolds.

__Defn__: Let X be a 4-manifold. A (g, k_1, k_2, k_3)-trisection of X is a decomposition: