Trisections are a decomposition of 4-manifolds that take inspiration from Heegaard splittings. Let's talk about this first.

**Definition:** A Heegaard splitting of $M^3$ is a decomposition $M = H_+ \cup H_-$ where $H_+ = S^1 \times B^2$, $H_- = S^1$.

(Recall: $D(X \times Y) = D X \# D Y$)

There's a proof of existence:

$\rightarrow$ 3-manifolds are triangulable [Moise, Bing 1952] but this is a hard fact that depends on $\dim M = 3$.

A regular nbhd of the 1-skeleton is a handlebody; its complement is also a handlebody since this is a regular nbhd of the dual simplex.

$\rightarrow$ It's not entirely obvious this is smooth, but it's true. $\square$

We will give another proof that is more adapted to higher
We first need some Morse theory.

**Defn**: A function $f: M \to \mathbb{R}$ is called Morse if it has no degenerate critical points.

(Recall: degenerate means $\text{hess } f_p$ is singular)

**Moral**: generic functions are Morse: they're dense (open) in the set of smooth functions.

The index of a critical point is $\text{index } (\text{Hess } f)_p$; i.e. the number of negative eigenvalues.

**Morse Lemma**: let $f: M \to \mathbb{R}$, $p \in M$ be a non-deg. critical point. Then in coordinate charts at, $f = x_1^2 + \ldots + x_k^2 - x_{k+1}^2 - \ldots - x_n^2$ ($f$ has index $k$)
Proof is bare hands, multivariable calc.

Remarks:
- critical pts are isolated.
- want to move/cancel critical pts.

Theorem: Let $X$ be a cascade from $M_0$ to $M_1$, and $f : X \to \mathbb{R}$ be Morse & have no critical points. Then $X \cong M_0 \times I$; in particular $M_0 \cong M_1$.

Proof: Pick $q$. Then $\Phi : X \to M_0 \times I$ is defined by flow along $V_q f$. This is a diffeo by ODE theorem.

Moreover, $f$ transforms to $\text{proj}_1$ under $\Phi$.

Take $W = (df)^\#$; then set $v = \frac{W}{\|W\|}$. Then $\|v\| = 1$.

$$df(v) = g(W, v) \equiv \frac{1}{\|W\|} g(W, W) \equiv 1.$$
\( \phi(t,p) = " \text{flow along } V \text{ by time } t \text{ from } p. " \)

Details to check, but this works. \( \square \)

\( \rightarrow \text{Moral:} \) If we don't see critical pts, topology is a product. When there is a critical pt, topology changes.

**Theorem:** Let \( f: X \to \mathbb{R} \) have a single critical point of index \( k \). Then \( X \cong (M_o \times I) \cup \{k\text{-handle}\} \).

**Proof:** Picture. \( h_k = D^k \times S^{n-k} \), attached along \( S^k \times S^{n-k} \) comp.

\( \rightarrow \) 1-handle: \( D^1 \times D^0 \) attached along \( S^0 \times S^0 \), etc.

\[ h_1 = D^1 \times D^0 \text{ attached via } S^0 \times D^1. \]
Moral: Morse functions give you a CW-decay.

Theorem: Let \( f : X \to R \) Morse with critical points of index \( k, l \) st. \( k < l \). Then we can slide the index \( k \) peak beneath the index \( l \) one.

\[
\text{ascending/descending manifolds intersect transversely in } f^{-1}(b).
\]

With the right dimension count, this means not at all.

Moral: We can arrange the critical points for \( f \) to be in increasing order.

Back to Heegaard splittings:

\[
\begin{array}{c}
\text{3} \\
\text{2} \\
\text{1} \\
\text{0}
\end{array}
\quad
\begin{array}{c}
\{3\} \\
\{2\} \\
\{1\}
\end{array}
\]

\((\text{Same } \# \text{ of } 1\text{s as } 2\text{s})\)
\[ f^{-1}(\frac{3}{2}) = \Sigma g; \quad \text{Moreover,} \quad f^{-1}(\frac{3}{2}, \frac{1}{2}) \cong \text{He} \quad \cong f^{-1}(\frac{1}{2}, \frac{1}{2}) \]

Which is the desired decomposition! Great.

\[ \text{Heegaard splittings can equiv. be thought of as taking } \Sigma \times I \text{ and attaching } 2\text{-handles in both directions.} \]

I.e., \( D^2 \times I^1 \) along \( S^1 \times S^1 \); thickened curves.

Claim: isotopy class of embeddings doesn't matter

Claim: Cop entirely with \( B^3 \)'s. [Cerf theorem]

With these: diagrams!

\[ \text{Theorem [Reidemeister-Singer]} \quad \text{Every 3-manifold admits a Heegaard splitting. Moreover any two given splittings are stably isotopic. (correct sum w/ } S^3 \text{)} \]
Diagrammatically, this means any two diagrams are hand slide-diffeo equivalent eventually.

**Theorem:** [Waldhausen] Every splitting of $S^3$ is stabilized.

**Theorem:** [Haken's Lemma] Heegaard splittings of reducible manifolds are reducible.

**Remarks:** Heegaard splittings/diagrams are useful for various things:
- Heegaard-Floer homology
- Minimal surfaces are $H^3$?

**Note:** This means we can't bisect 4-manifolds.

**Definition:** Let $X$ be a 4-manifold. A $(g, k_1, k_2, k_3)$-trisection of $X$ is a decomposition: