Trisections & Heegard splittings

(Notes by S. Dwivedi)

- Heegard Splittings ↳ Morse Theory
- Diagrams
- Trisections

Eventual goal - reach trisections
↳ 4 dim analog of something 3-dimensional

Expository talk.

Def^n:- Let M^3 be a 3-manifold. A Heegard splitting of M is $M^3 = H_g^3 \cup_{\Sigma_g} H_g^3$; where

$H_g = \natural^g S^1 \times B^2$; $\partial H_g = \#^g S^1 \times S^1 = \Sigma_g$ ↳ surface of genus g

$\partial(X \# Y) = \partial X \# \partial Y$

$g \rightsquigarrow$ genus of the handle body H_g

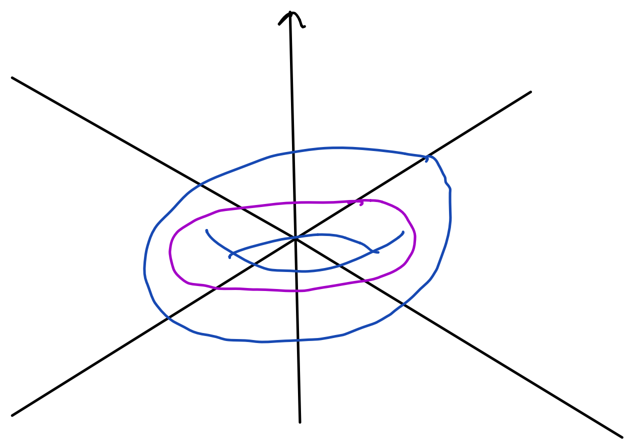
$\natural^g \rightsquigarrow$ Boundary connected sum

Examples

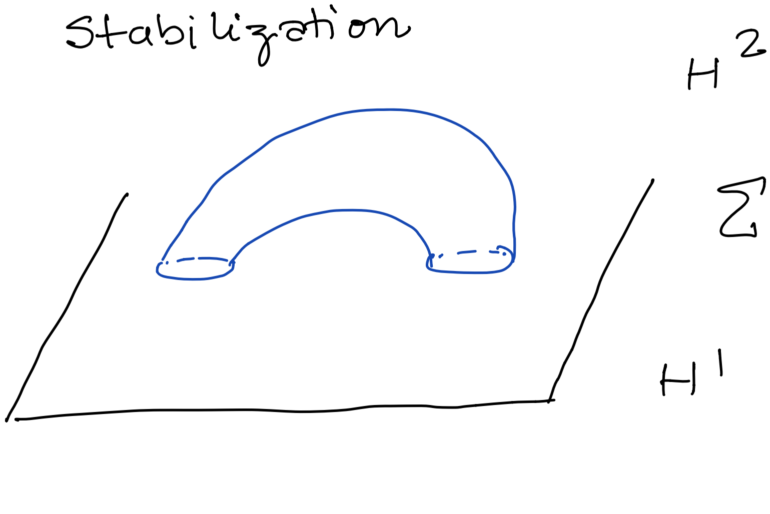
(1) $S^3 = B^3 \cup_{S^2} B^3$ $\quad (S^2 = B^2 \cup_{S^1} B^2)$

↳ genus zero

(2) $S^3 = H_1 \cup H_1$, H_1 = solid torus



3) Stabilization



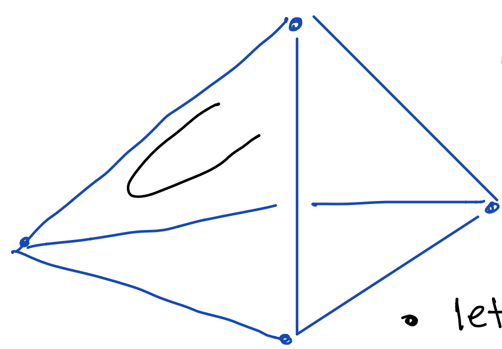
from stabilization :- If we have a Heegard splitting of genus g then it has a Heegard splitting of all higher genus.

<u>Theorem</u> :- Given M^3, $\exists$ Heegard splitting of genus g for some g.

<u>Proof</u>

→ 3 manifolds are triangulizable [Moise, Bing 50's]

→ Fix $M \simeq K$ → simplicial complex



• let $H^1 = v(K^{(1)}) \cong H_g$ (some g)

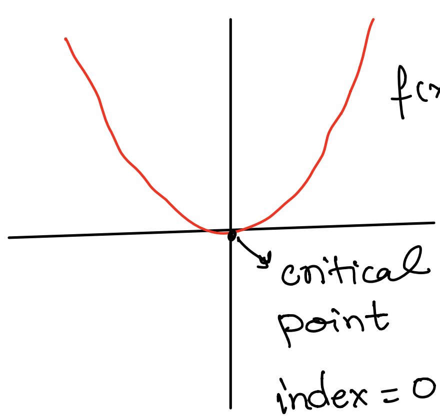
• let $H^2 = M \setminus v(K^{(1)}) = v(\text{dual to } K^{(1)})$

Then $M = H_g \cup_{\Sigma_g} H_g$ ∎

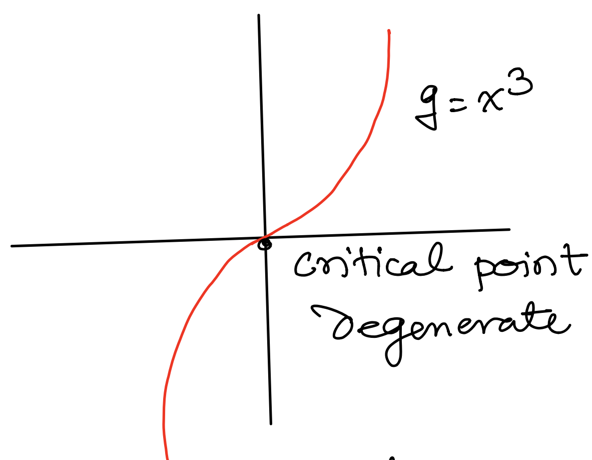
2nd proof :→

Def^n :→ A function $f: M \to \mathbb{R}$ is called **Morse** if f has no degenerate critical points where degenerate $\iff$ $\text{Hess} f \big|_{pt}$ is singular.

- The index of a critical point is the index of $\text{Hess} f$ = # of negative eigenvalues.



Morse



Not Morse

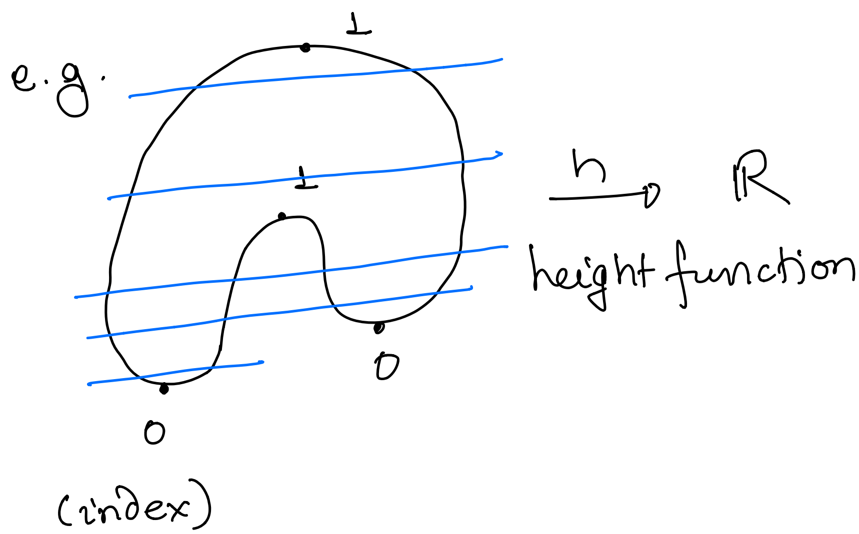
$h = x^3 - tx$ is Morse

There are lots of Morse functions, they are generic & stable.

Morse Lemma Let $f: M \to \mathbb{R}$ be Morse and p be a critical point (index k). Then there are coordinates s.t.

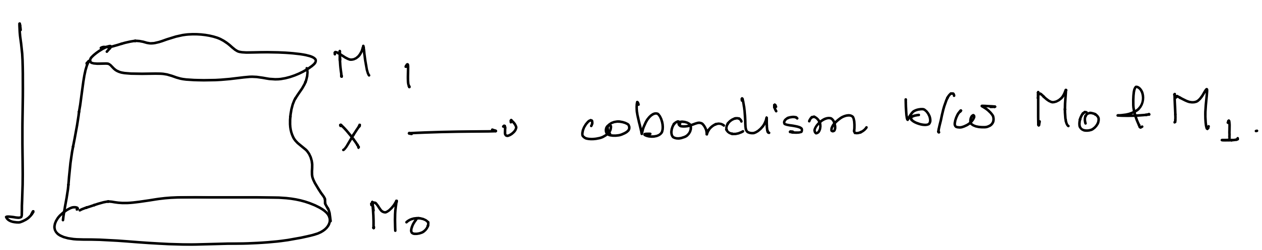
$$f = -x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2$$

e.g.



Prop:- Let $f: X \to \mathbb{R}$ be Morse w/ no critical points (X a cobordism from $M_0 \to M_1$)

Then $X \cong M_0 \times I$ $(M_0 \cong M_1)$.



$\downarrow$ so the topology is not changing outside a nbd of critical points.

Proof [Sketch]

- pick g
- Look at $\nabla_g f = v$
- $df(v) = 1$

$$\Phi : M_0 \times I \longrightarrow X$$

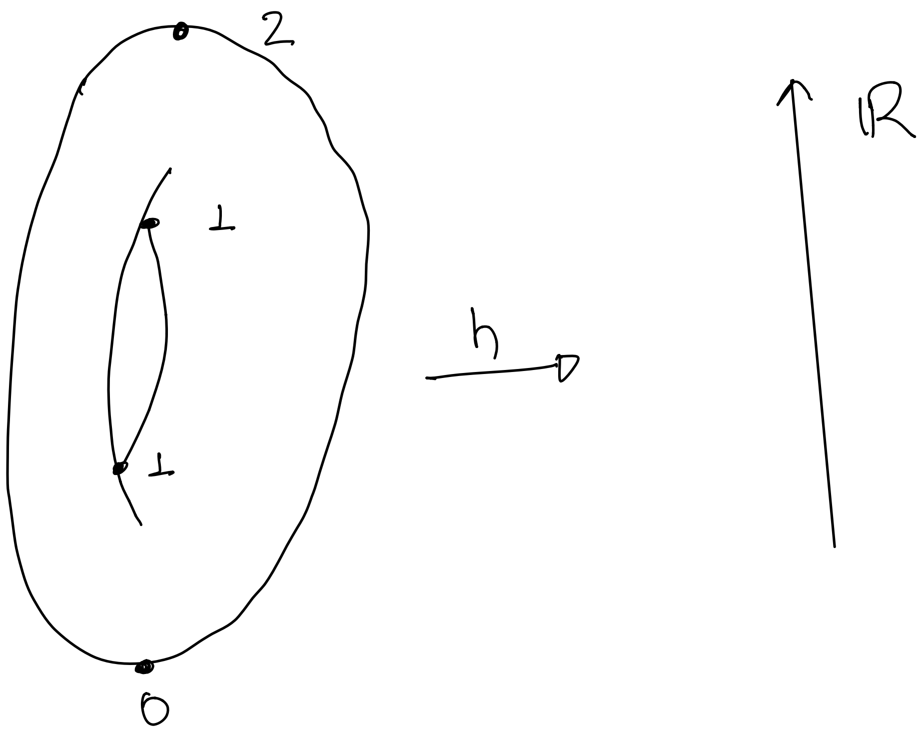
$(p, t) \longmapsto$ "flow p by v for time t".

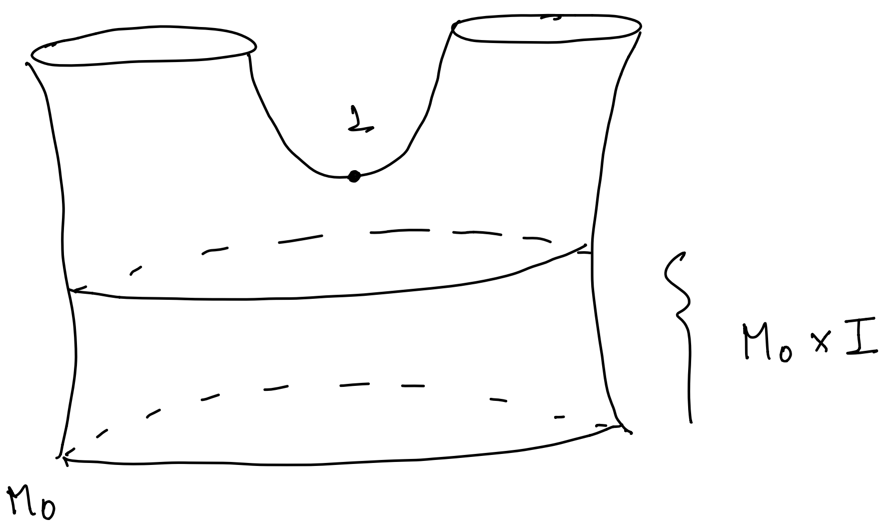
Proposition :- Let $f : X \to \mathbb{R}$ be Morse w/ one critical pt w/ index k. Then

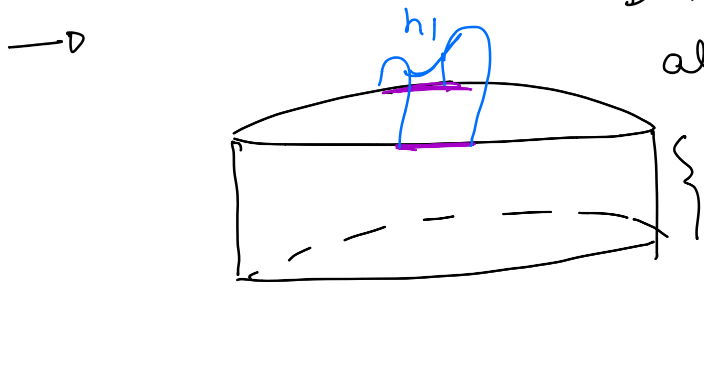
$$X \cong (M_0 \times I) \cup \{ k\text{-handle} \}$$

$h_k = D^k \times D^{n-k}$ attached along $S^{k-1} \times D^k$

$$\partial h_k = S^{k-1} \times D^{n-k} \cup D^k \times S^{n-k-1}$$

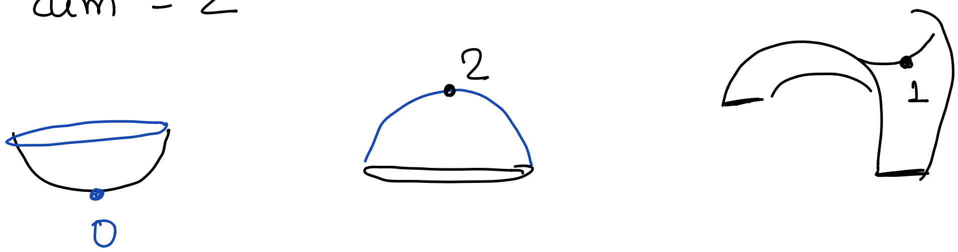




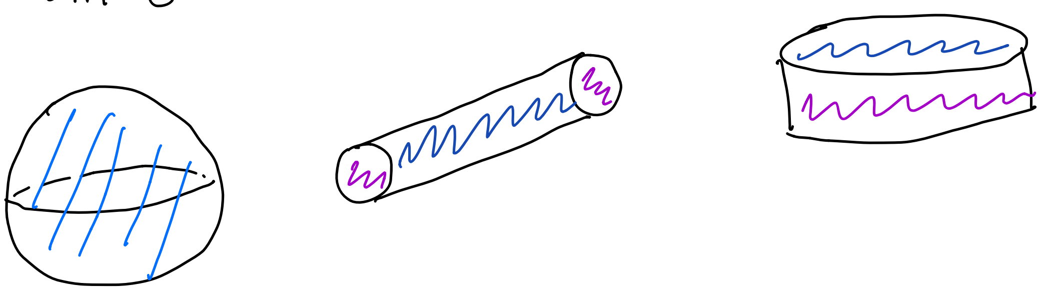


$D^1 \times D^1$ attached along $S^0 \times D^1$

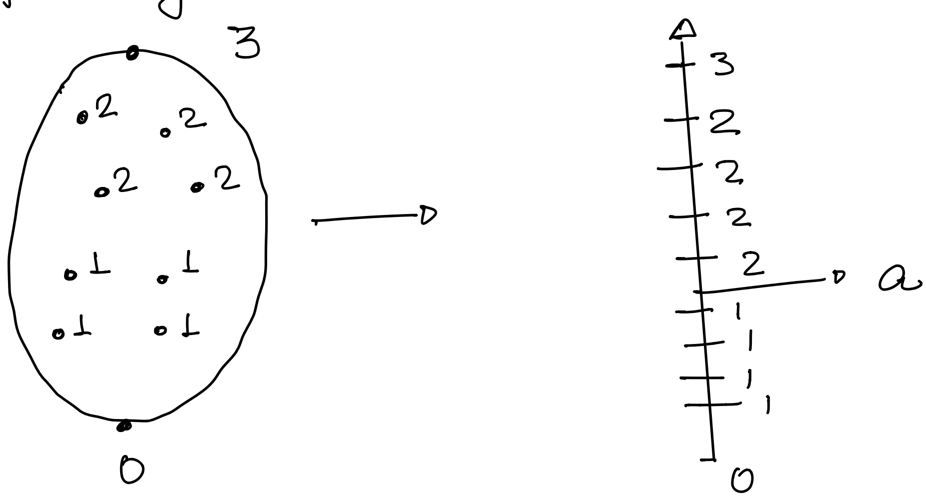
dim = 2



dim 3



Fact :→ for M^3 we can arrange the following



$f^{-1}(a) \cong \Sigma_g$ for some g as "a" is a regular value.

$$f^{-1}((-\infty, a]) \cong H_g$$

$$f^{-1}([a, \infty)) \cong H_g$$

<u>Claim</u> :→ We can build M "inside out"

(3-h)

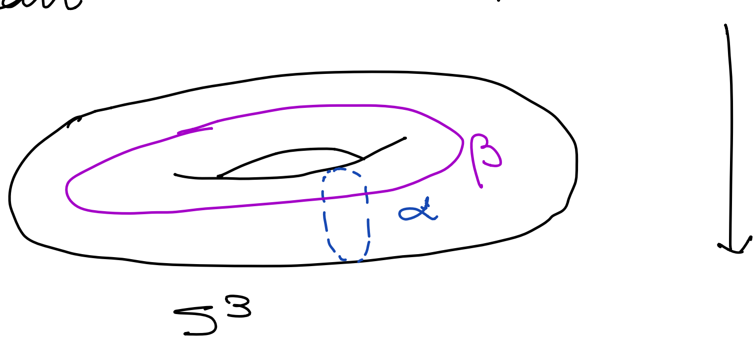
↑ 2-h

$\Sigma \times I$

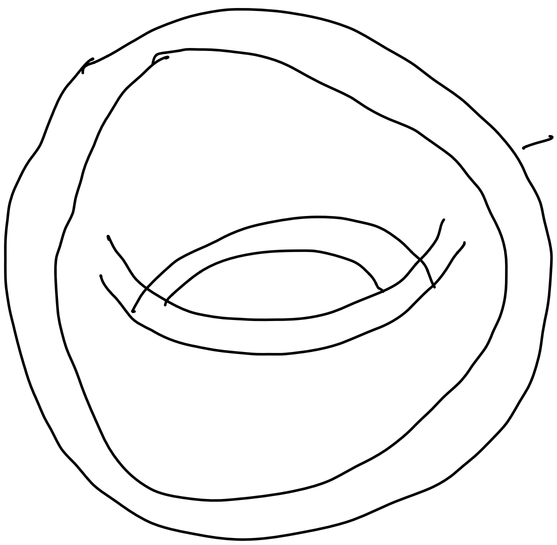
↓ 2-h

(3-h)

We can describe M w/ a diagram.

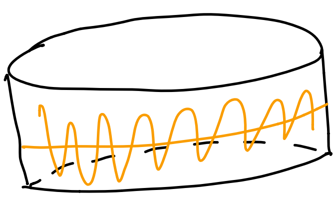


Here's how :-



$\partial = \Sigma \times \{0\} = \alpha$
$\cup\ \Sigma \times \{1\} = \beta$

- Start w/ $\Sigma \times I$.
- Attach 2-handles to $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$ via α, β.



glue so that c gets sent to α.

- Then "fill" w/ 3-handles
- Attach 2-handles to $\Sigma \times \{1\}$ and glue so that the core gets sent to β.

Again "fill" w/ 3-handle.

Imagine having a donut and fill the gap w/ a ball $\rightsquigarrow$ kill the genus.

The original $\rightsquigarrow$ $B^3 \cup_{S^2} B^3$

$\rightsquigarrow$ given M^3, $\exists$ a Heegard splitting of certain genus. (no lower bound for genus g).

<u>Theorem</u> :- [Laudenbach - Poénaru]

Any diffeomorphism of $\#^k S^1 \times S^2$ extends across $\natural^k S^1 \times B^3$.