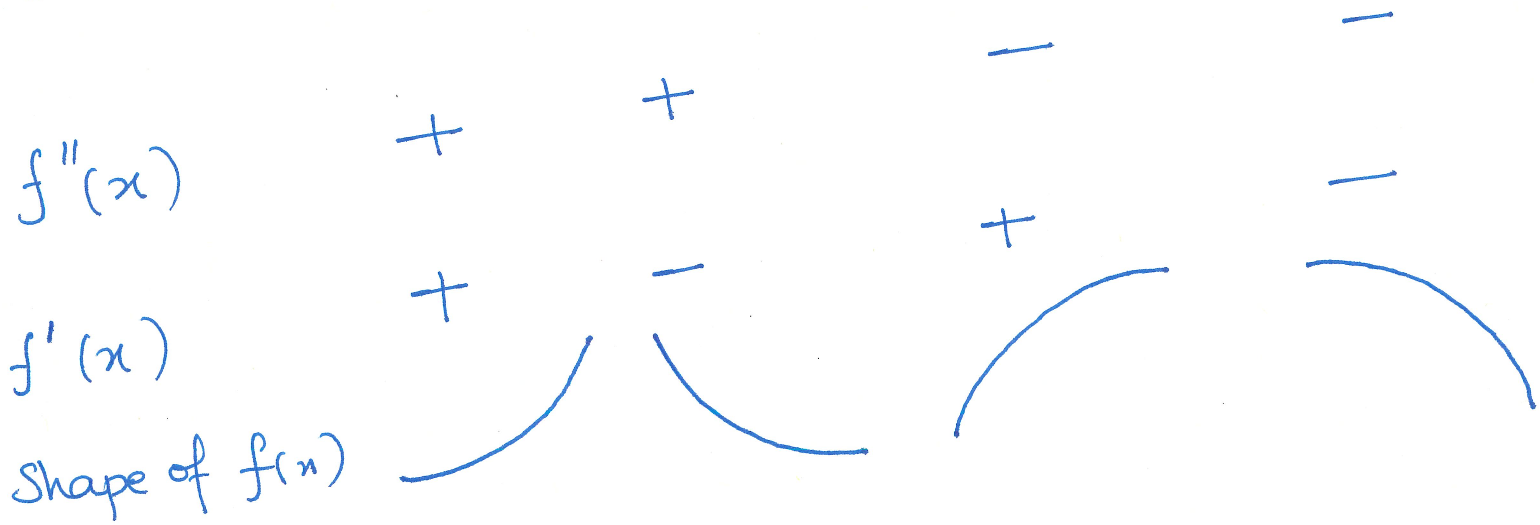
Curve Sketching (Using Calculus)

$f(x)$ be a function.

1. find the domain of $f(x)$.
2. find the y-intercept $(x=0)$ and x-intercepts ($y=0$ and solve for x)
3. find vertical asymptotes. look for points x where there is a division by zero. find horizontal asymptotes. Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.
4. Calculate all the critical points. ($f'(x)=0$ or DNE) look for local max/min and the intervals where $f(x)$ is increasing/decreasing.
5. Calculate inflection points. ($f''(x)=0$ or DNE) Look for intervals where $f(x)$ is concave up/down.

⑥ Make a table of all the points in Steps 1-5. Plot these points on the graph.

⑦ Join the above points as follows:-



Ques: Draw the graph of $f(x) = x^3 - 6x^2 + 9x$.

Sol ① $f(x)$ is a polynomial $\Rightarrow$ domain $= \mathbb{R}$ or $(-\infty, \infty)$

② y-intercept $\quad x = 0 \Rightarrow$

$$f(0) = 0^3 - 6.0^2 + 9.0 = 0$$

$$\boxed{(0,0)}$$

x-intercept

$$y = 0 \Rightarrow x^3 - 6x^2 + 9x = 0$$

$$\Rightarrow x(x^2 - 6x + 9) = 0$$

$$\Rightarrow x(x-3)^2 = 0$$

$$\therefore \quad x = 0 \quad \text{or} \quad x = 3$$

$\boxed{(0,0) \text{ and } (3,0) \text{ are the } x\text{-intercepts.}}$

(3) vertical asymptotes :- none

horizontal asymptotes :- $\lim_{x \to \infty} f(x) = \infty$

$\lim_{x \to -\infty} f(x) = \pm\infty$

There are no horizontal asymptotes.

(4) critical points.

$$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3).$$

$f'(x)$ exists everywhere.

$$f'(x) = 0 \Rightarrow 3(x^2 - 4x + 3) = 0$$

$$\Rightarrow 3(x-3)(x-1) = 0$$

$$\Rightarrow x = 1 \quad \text{or} \quad x = 3$$

$\boxed{(1,4) \text{ and } (3,0) \text{ are the critical points.}}$

(5) $f''(x) = 6x - 12 = 6(x-2)$

$f''(x) = 0 \Rightarrow 6(x-2) = 0 \Rightarrow x = 2$

$\therefore$ $\boxed{(2,2) \text{ is an inflection point.}}$

at $x = 1$, $f''(1) = 6(1-2) = -6 < 0$

$\Rightarrow$ $x = 1$ is a local max.

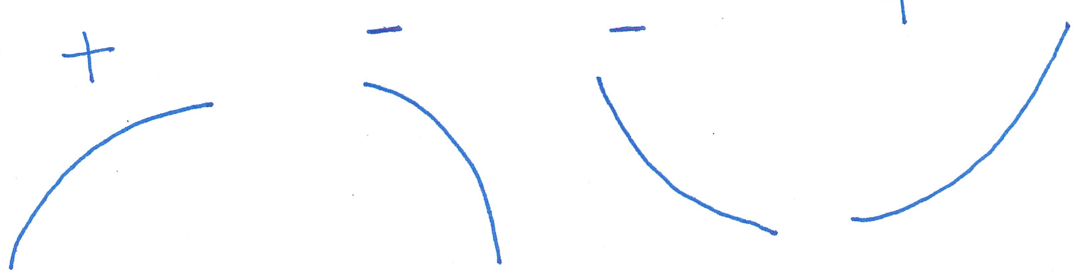
$x = 3$, $f''(3) = 6(3-2) = 6 > 0$

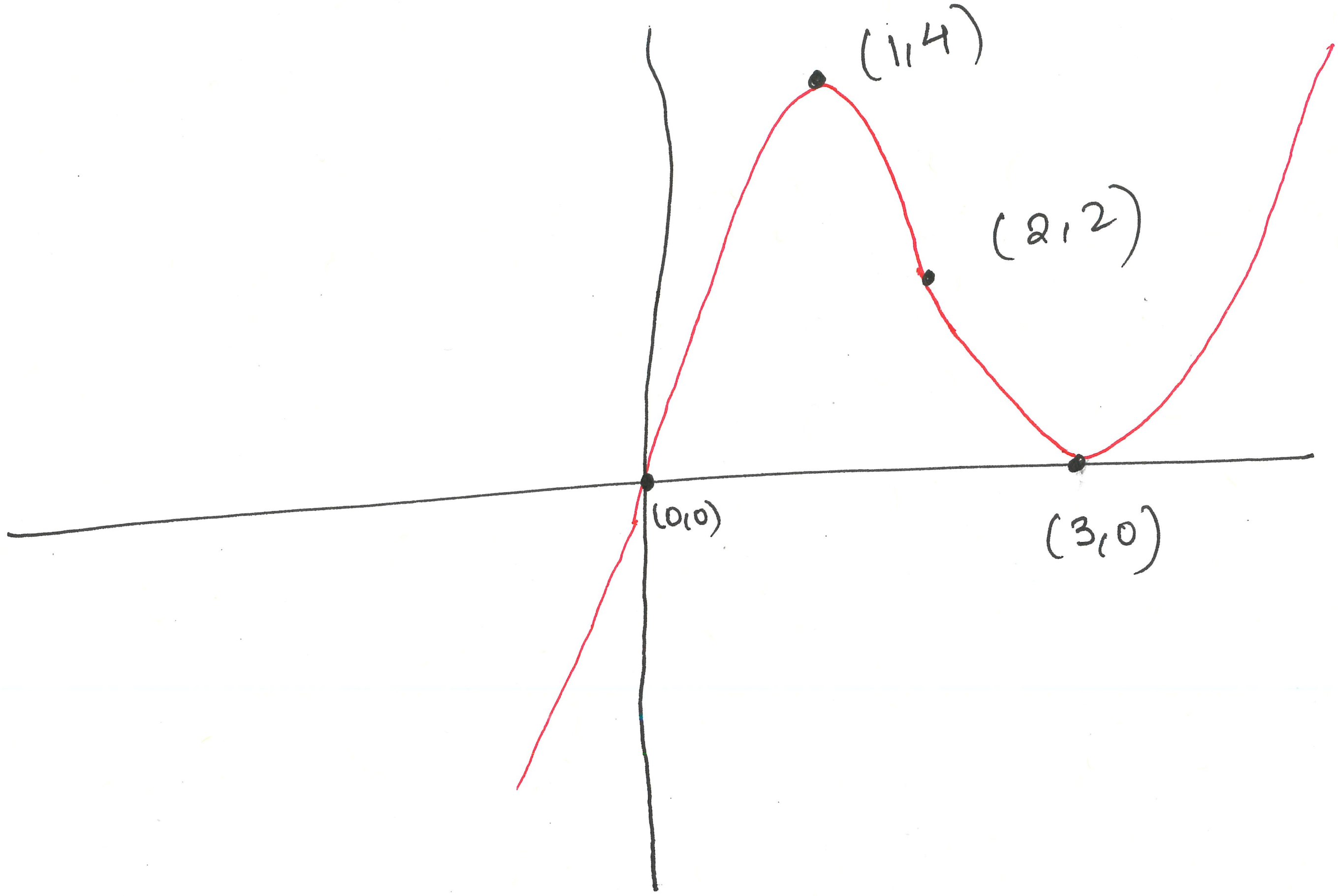
$\Rightarrow$ $x = 3$ is a local min.

(6)

		1		2		3	
$f''(x)$	$-$		$-$		$+$		$+$
$f'(x)$	$+$		$-$		$-$		$+$

shape of $f(x)$





final Graph

Ques :- Draw the graph of $f(x) = \dfrac{x^2}{x^2-4}$.

Sol^n (1) $f(x) = \dfrac{x^2}{(x+2)(x-2)}$

$\Rightarrow$ Domain : $\mathbb{R} \setminus \{2,-2\}$ or $\{x \in \mathbb{R} \mid x \neq \pm 2\}$

(2) y - intercept $x = 0 \Rightarrow y = 0$

$\Rightarrow$ $\boxed{(0,0)}$ - y-intercept.

x - intercept $y = 0 \Rightarrow \dfrac{x^2}{x^2-4} = 0$

$\Rightarrow x = 0$

$\therefore$ $\boxed{(0,0)}$ - x-intercept.

(3) Asymptotes

vertical :- possible points for vertical asymptotes are $x = 2$ and $x = -2$

$x = 2$ and -2 give non-zero numerator.

$\Rightarrow$ $\boxed{\text{vertical asymptotes at } x = \pm 2}$

horizontal : $\lim_{x \to \infty} f(x)$ $\qquad \lim_{x \to -\infty} f(x) = 1$

$$\Rightarrow \lim_{x \to \infty} \frac{x^2}{x^2-4} = \lim_{x \to \infty} \frac{\cancel{x^2}}{\cancel{x^2}\left(1-\frac{4}{x^2}\right)}$$

$$= 1$$

$\Rightarrow$ $x = 1$ is a horizontal asymptote.

(4) Critical points

$$f'(x) = \frac{(x^2)' \cdot (x^2-4) - (x^2-4)' \cdot x^2}{(x^2-4)^2}$$

(quotient rule)

$$= \frac{2x(x^2-4) - (2x \cdot x^2)}{(x^2-4)^2}$$

$$= \frac{-8x}{(x^2-4)^2}$$

$f'(x)$ DNE at $\underbrace{x = \pm 2}$ and $f'(x) = 0$ at $x = 0$

are not critical points as they are not in the domain.

$\therefore$ $(0,0)$ is the only critical point.

(5)

$$f''(x) = \frac{(-8x)'(x^2-4)^2 - ((x^2-4)^2)' \cdot (-8x)}{((x^2-4)^2)^2}$$

$$= \frac{-8(x^2-4)^2 + 2(x^2-4) \cdot 2x \cdot 8x}{((x^2-4)^2)^2}$$

$$= \frac{\cancel{(x^2-4)}\left(-8(x^2-4) + 32x^2\right)}{(x^2-4)^{\cancel{4}\,3}}$$

$$= \frac{24x^2+32}{(x^2-4)^3} = \frac{8(3x^2+4)}{(x^2-4)^3}.$$

$f''(x)$ DNE at $x = \pm 2 \rightsquigarrow$ not an inflection point.

$f''(x)$ is never zero as $8(3x^2+4) \neq 0$.

$$f''(0) = \frac{8(3.0^2+4)}{(0^2-4)^3} < 0$$

$\Rightarrow (0,0)$ is local max.

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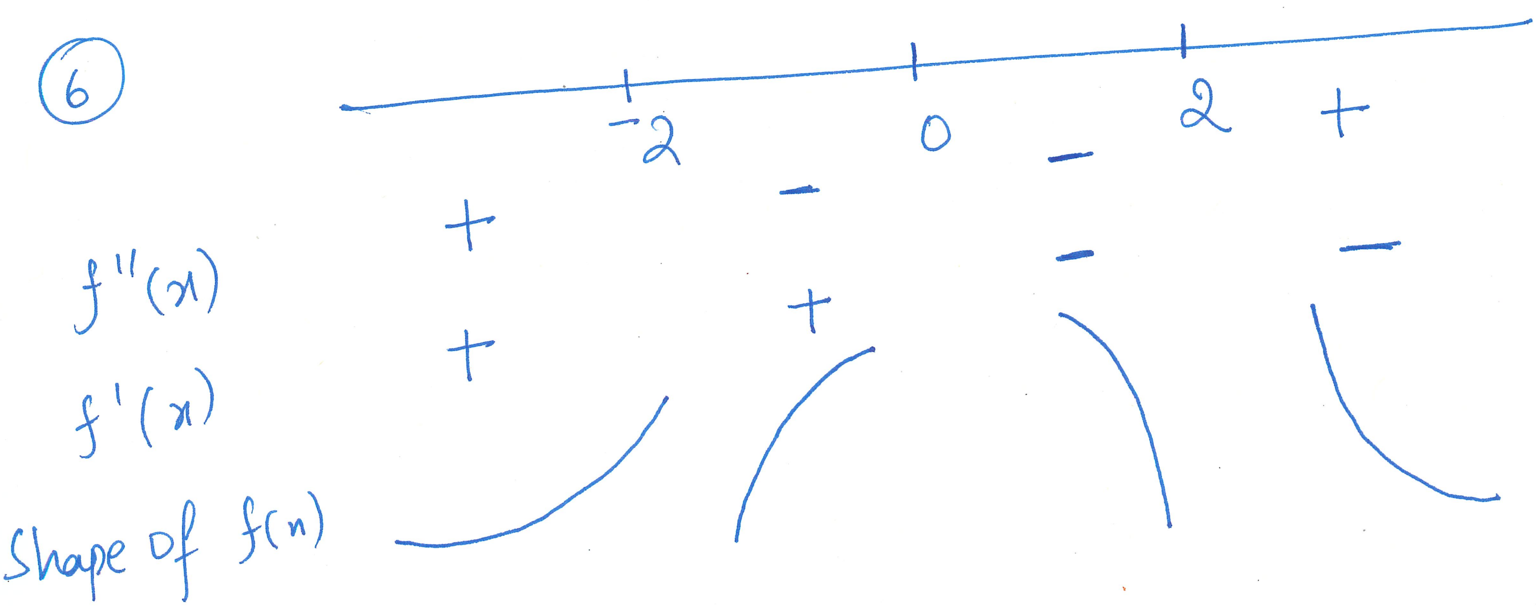

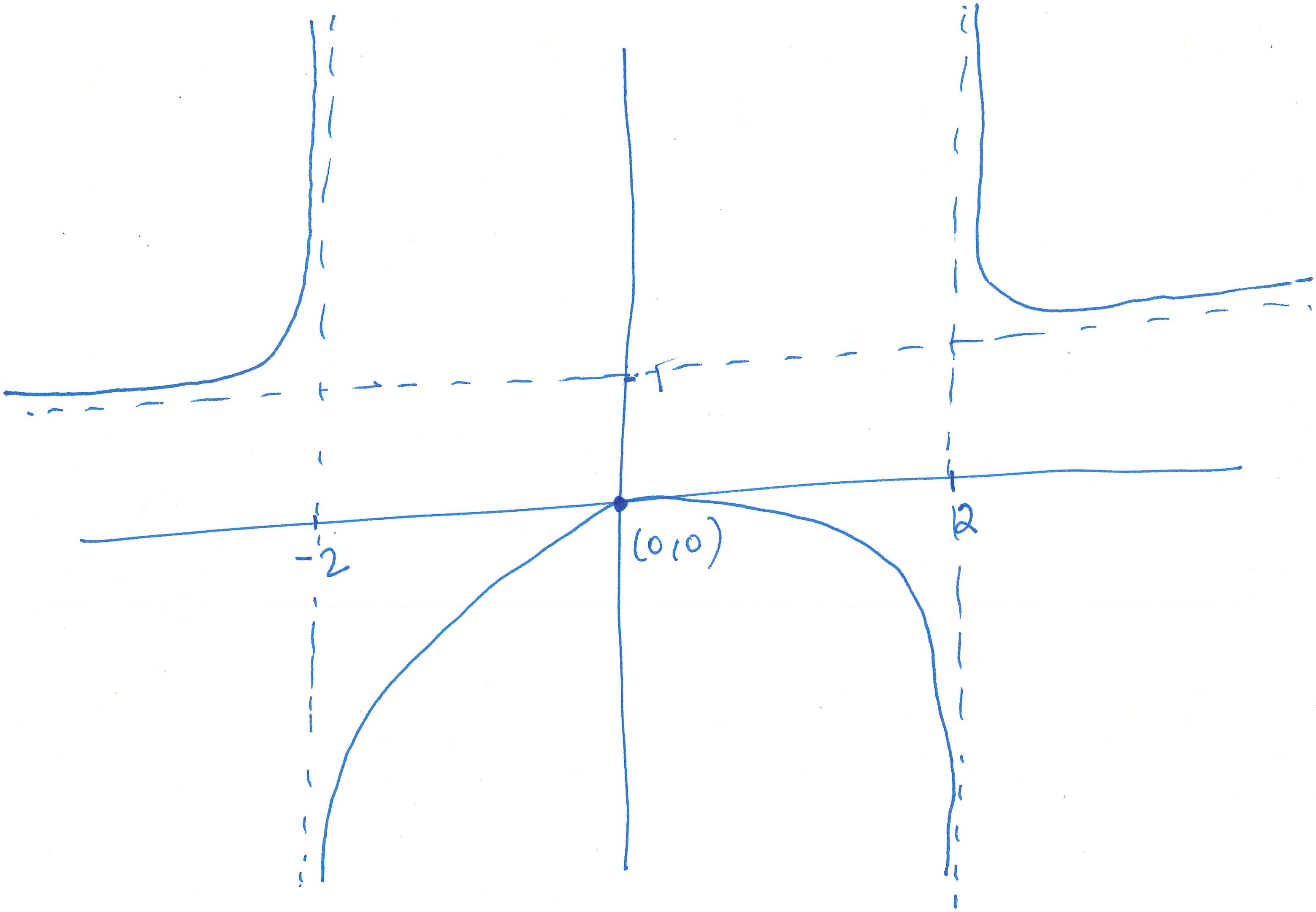

final graph..

This marks the end of differential calculus.

from the next lecture, we'll start with integral calculus.

o——————x——————x——————o