Lecture 27

Related Rates

The idea for solving related rates problems is the following:- suppose x and y are related to each other. If both of them are changing according to time t and we know $\frac{dx}{dt}$, then using the relation between x and y, we can find $\frac{dy}{dt}$.

The strategy for solving problems here is same as that in optimization.

① Draw a figure of the situation in question and use variables to represent any relevant quantities.

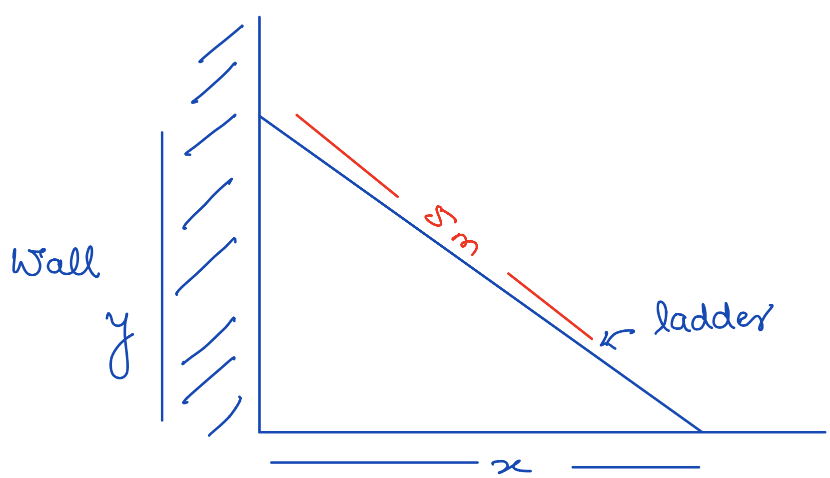
② find an equation or relation which related the variables in ①. List any other information from the question.

③ Differentiate the relation in ② with respect to time t. You'll have to use implicit differentiation.

④ Solve for the unknown quantity.

Ques :- A 5 m long ladder sits against a vertical wall. If the bottom of the ladder is slipping away at the rate of 1 m/s, how fast is the top of the ladder slipping down the wall. when the bottom is 4 m away?

Sol^n We draw a figure.



Let us denote by x = distance of the bottom of the ladder from the wall

y = distance of the top of the ladder from the bottom of the wall.

Then we have a right-angled triangle

$\Rightarrow \quad x^2 + y^2 = 5^2 \Rightarrow \boxed{x^2+y^2 = 25.}$ — ①

This is the relation b/w x and y from the question.

given :- $\dfrac{dx}{dt} = 1 m/s$

Ques :- $\dfrac{dy}{dt} = ?$ when $x = 4$ m

when $x = 4 \Rightarrow 4^2 + y^2 = 25$

$\Rightarrow y^2 = 9 \Rightarrow y = +3.$

To find $\dfrac{dy}{dt}$, we differentiate (implicity) ① to get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

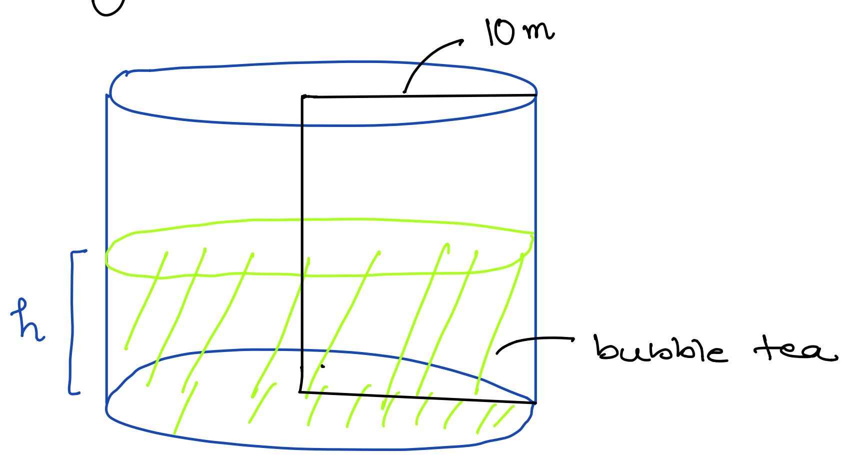
$$\Rightarrow \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = -\frac{4}{3}\cdot 1$$

$$= -\frac{4}{3} \text{ m/s}$$

$\therefore$ The top of the ladder is slipping down at $\frac{4}{3}$ m/s

Ques A cylindric vessel of radius 10 m is being filled by bubble tea. If the height of the bubble tea is increasing at 2 m/s, how fast is the volume of the bubble tea increasing?

Soln



The situation is shown in figure above. Let

h = height of the bubble tea

Then volume of the bubble tea, $V = \pi r^2 h$

$$= 100\pi h \quad (\text{as } r = 10) \longrightarrow (1)$$

so we got the relation.

$\frac{dh}{dt} = 2\, m/s$ is given.

Want $\frac{dV}{dt}$. So let's differentiate (i) w.r.t. t.

to get

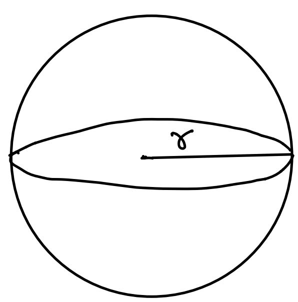
$$\frac{dV}{dt} = 100\pi \frac{dh}{dt} = 100\pi \cdot 2$$

$$= 200\pi\ m^3/s$$

$\therefore$ the volume of the bubble tea is increasing at $200\pi\ m^3/s$.

<u>Ques.</u> Air is leaking from a hot air balloon. If the radius of the balloon is changing at 0.5 m/s, then how fast is the volume of the balloon decreasing when the radius is 1m?

<u>Soln</u> The figure is as follows.



$r = 1m$

If V = volume of the balloon then

$V = \frac{4}{3}\pi r^3$ (volume of a sphere as the balloon is spherical). —— ①

This is the relation

$\frac{dr}{dt} = -0.5$ m/s. (− as the radius is decreasing with t).

Want $\frac{dV}{dt}$ when $r = 1$ m.

We differentiate ① to get

$$\frac{dV}{dt} = 3 \cdot \frac{4}{3}\pi r^2 \cdot \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$= 4\pi (1)^2 \cdot (-0.5)$$

$$= -2\pi \ m^3/s$$

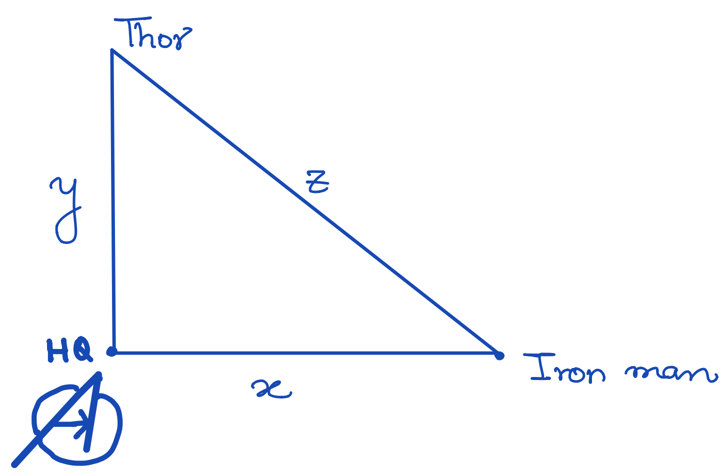
∴ the volume of the balloon is decreasing at the rate of $2\pi \ m^3/s$.

Ques. Iron man leaves the Avengers HQ at 12 noon, travelling east at 100 km/h.

Thor, the god of thunder, leaves the Avengers HQ at 3 PM, travelling north at the speed of 150 km/h.

How fast is the distance b/w them changing at 5 PM?

Soln The situation is as follows.



Let t = time in hours

x = distance in km, covered by Iron man at time t.

y = distance in km, covered by Thor at time t.

z = distance in km b/w them at time t

note that $\underbrace{z^2 = x^2 + y^2}_{\text{relation}}$ —— (1)

$$\frac{dx}{dt} = 100 \text{ km/h} \quad , \quad \frac{dy}{dt} = 150 \text{ km/h}$$

Want : $\frac{dz}{dt}$ at 5 PM.

At 5 PM, $x = 100 \cdot 5 = 500$ km as Iron man has been travelling for 5 hrs.

Thor has only been travelling for 2 hrs (he started at 3 PM) $\Rightarrow$ $y = 150 \cdot 2 = 300$ km

$$\Rightarrow \quad z^2 = x^2 + y^2 = 500^2 + 300^2 = 3400$$

$$\Rightarrow \quad z = 100\sqrt{34} \text{ km.}$$

now we differentiate ① to get

$$2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$

$$\Rightarrow \quad \frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt} \right)$$

$$= \frac{1}{100\sqrt{34}}\left(500 \cdot 100 + 300 \cdot 150 \right)$$

$$\Rightarrow \quad \frac{dz}{dt} = \frac{950}{\sqrt{34}} \text{ Km/h}$$

$\therefore$ The distance b/w Iron man and Thor is changing at $\dfrac{950}{\sqrt{34}}$ km/h.

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