Lecture 23

In this lecture, we will use the derivative of $f(x)$ to determine the intervals where the function is incre-asing or decreasing.

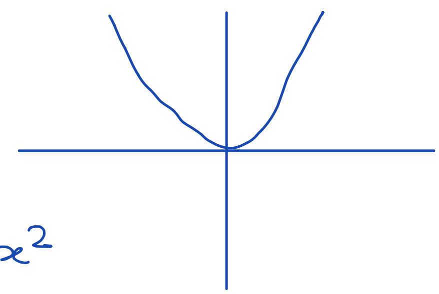
Suppose $f(x)$ is a function and $f'(x)$ exists on an interval I.

- If $f'(x) > 0 \ \forall\, x \in I$, then $f(x)$ is increasing on I.
- If $f'(x) < 0 \ \forall\, x \in I$, then $f(x)$ is decreasing on I.

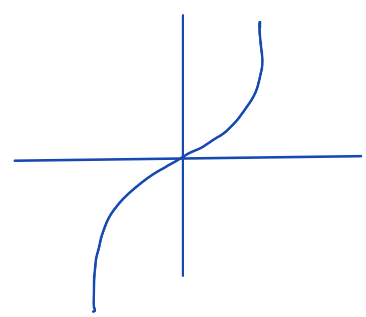
- If you recall the graph given in Lec. 22, then you will observe that wherever the function was increasing, $f'(x)$ (or slope of the tangent line) was positive and $f'(x) < 0$ for the intervals where $f(x)$ was decreasing.

e.g. 1) $f(x) = x^2 \Rightarrow f'(x) = 2x$

$f'(x) > 0$ on $(0,\infty) \Rightarrow f(x)$ is increasing on $(0,\infty)$ and $f'(x) < 0$ on $(-\infty, 0) \Rightarrow f(x)$ is decreasing on $(-\infty, 0)$. This can be seen from the graph as well.



ii) If $f(x) = x^3 \Rightarrow f'(x) = 3x^2$ which is always positive. Thus $f(x) = x^3$ is increasing on $\mathbb{R}$.



We saw in the example in Lec. 22 that the function was changing its behaviour at a point where either $f'(x) = 0$ or $f'(x)$ DNE or $f(x)$ DNE.

<u>Def</u> A <u>critical point (or critical number)</u> of

$f(x)$ is a point c in the domain such that

1) $f'(c) = 0$ or

2) $f'(c)$ DNE.

Remark :- In order for a point to be a critical point, it must be in the domain of the function.

The reason for the importance of critical points is that $f(x)$ can go from increasing to decreasing (or vice-versa) only at a critical point or at a point where $f(x)$ is undefined.

Thus we have the following test for finding the intervals where a function is increasing/decreasing.

Test for increasing/decreasing.

Given a function $f(x)$

1. Find $f'(x)$.

2. Find all critical points by equating $f'(x)=0$ and also the points where $f'(x)$ DNE.

③ Make sure that the critical points in step ② are all in the domain, otherwise discard them.

④ Plot the critical points and the points where $f(x)$ is undefined on a real line.

⑤ Check where $f'(x) > 0 \rightsquigarrow f(x)$ increasing.

$f'(x) < 0 \rightsquigarrow f(x)$ decreasing.

<u>Ques</u> where is $f(x) = 2x^3 + 15x^2 - 36x + 10$ increasing / decreasing.

<u>Soln</u> ① $f'(x) = 6x^2 + 30x - 36$

which is a polynomial and hence exists everywhere.

② $f'(x) = 0 \Rightarrow 6x^2 + 30x - 36 = 0$

$\Rightarrow 6(x^2 + 5x - 6) = 0$

$\Rightarrow 6(x-3)(x-2) = 0 \Rightarrow x = 3$ or $x = 2$

Both these points are in the domain $\Rightarrow$ critical points are $x = 3$ and $x = 2$.

③

$+$		$-$		$+$
$f'(x)>0$	2	$f'(x)<0$	3	$f'(x)>0$

④ Thus $f(x)$ is increasing on $(-\infty, 2) \cup (3, \infty)$

decreasing on $(2, 3)$.

<u>Ques</u> find where $f(x)$ is increasing/decreasing.

1) $f(x) = \dfrac{x+1}{x-2}$ 2) $f(x) = x^{2/3}$

<u>Sol</u> 1) 1) By quotient rule

$$f'(x) = \frac{(x+1)'(x-2) - (x+1)(x-2)'}{(x-2)^2}$$

$$= \frac{(x-2)-(x+1)}{(x-2)^2} = \frac{-3}{(x-2)^2}$$

2) $f'(x)$ is never zero. $f'(x)$ DNE at $x=2$ BUT

$x=2$ is NOT in the domain of $f(x)$.

Thus we discard it.

3) There are no critical points $\Rightarrow$ we only plot

$x=2$ on the real line as $f(x)$ is undefined there.

$$\begin{array}{ccc} - & & - \\ \hline & 2 & \end{array}$$

$\because \ f'(x) = \dfrac{-3}{\underbrace{(x-2)^2}_{\text{positive}}} < 0 \implies f(x)$ is decreasing on $(-\infty, 2) \cup (2, \infty)$.

Thus, there might be situations where there are no critical points.

2) $f(x) = x^{2/3}$

1) $f'(x) = \dfrac{2}{3x^{1/3}}$

2) $f'(x)$ is never zero. $f'(x)$ DNE at $x=0$ and this time $x=0$ is in the domain $\implies x=0$ is a critical point.

3) $f(x)$ is defined everywhere $\implies$ we only plot $x=0$.

$$\begin{array}{ccc} - & & + \\ \hline f'(x)<0 & 0 & f'(x)>0 \end{array}$$

Thus $f(x)$ is increasing in $(0, \infty)$ and decreasing in $(-\infty, 0)$.

In the next lecture, we will study maxima and minima.

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