Lecture 21

Derivatives of Trigonometric Functions

We'll need to use the trigonometric identities

$\sin^2 x + \cos^2 x = 1$

$\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x$

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$\tan x = \dfrac{\sin x}{\cos x}$, $\sec x = \dfrac{1}{\cos x}$, $\operatorname{cosec} x = \dfrac{1}{\sin x}$

Useful Fact

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1} \quad \text{— } (1)$$

Remark $\displaystyle\lim_{x \to \pi/2} \frac{\sin x}{x} \neq 1$, $\displaystyle\lim_{x \to \infty} \frac{\sin x}{x} \neq 1$

For $\cos x$

$$\boxed{\lim_{x \to 0} \frac{\cos x - 1}{x} = 0} \quad \text{— } (2)$$

To see eq. (2) using eq. (1), we observe that

$$\lim_{x\to 0} \frac{\cos x - 1}{x} = \lim_{x\to 0} \frac{\cos x - 1}{x} \cdot \frac{\cos x + 1}{\cos x + 1}$$

$$= \lim_{x\to 0} \frac{\cos^2 x - 1}{x \cdot (\cos x + 1)}$$

$$= \lim_{x\to 0} \frac{-\sin^2 x}{x(\cos x + 1)}$$

$$= \lim_{x\to 0} - \underbrace{\frac{\sin x}{x}}_{=1} \cdot \frac{\sin x}{(\cos x + 1)}$$

$$= \lim_{x\to 0} -1 \cdot \frac{\sin x}{\cos x + 1} \quad (\text{eq. ① here})$$

$$= 0 \quad (\text{put } x = 0)$$

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① $\boxed{\frac{d}{dx}(\sin x) = \cos x}$

To see how we get this, we use the definition of the derivative to calculate

$$\frac{d}{dx}(\sin x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

(Used the formula for $\sin(A+B)$.)

$$= \lim_{h \to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

$$= \lim_{h \to 0} \sin x \cdot \underbrace{\frac{(\cos h - 1)}{h}}_{=0 \ (\text{eq. } ②)} + \lim_{h \to 0} \underbrace{\frac{\cos x \sin h}{h}}_{=1 \ (\text{eq. } ①)}$$

$$= \sin x \cdot 0 + \cos x \cdot 1 = \cos x$$

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② $\frac{d}{dx}(\cos x) = -\sin x$

Same method as that for $\sin x$, but instead use the formula for $\cos(A+B)$.

③ find. $\frac{d}{dx}(\tan x)$.

$\tan x = \frac{\sin x}{\cos x}$ $\Rightarrow$ we can use the quotient rule.

$$\frac{d}{dx}(\tan x) = \frac{\frac{d}{dx}(\sin x).\cos x - \frac{d}{dx}(\cos x).\sin x}{\cos^2 x}$$

$$= \frac{\cos x.\cos x - (-\sin x).\sin x}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

$$\therefore \boxed{\frac{d}{dx}(\tan x) = \sec^2 x}$$

Exercise :- $\frac{d}{dx}(\cot x)$, $\frac{d}{dx}(\sec x)$ and $\frac{d}{dx}(\operatorname{cosec} x)$

Hint: Use the quotient rule and the formulas for $\cot x$, $\sec x$ and $\operatorname{cosec} x$.

Que. find the derivative of

(1) $f(x) = \ln(\sin x)$

(2) $f(x) = x^{\sin x}$

Sol. (1) $f(x) = g(h(x))$ where $g(x) = \ln x$

$h(x) = \sin x$

$\Rightarrow \frac{d}{dx} f(x) = g'(h(x)) \cdot h'(x)$, $g'(x) = \frac{1}{x}$

$h'(x) = \cos x$

$= \frac{1}{\sin x} \cdot \cos x = \boxed{\cot x}$

(2) logarithmic Differentiation.

$\ln f(x) = \ln\left(x^{\sin x}\right) = \sin x \cdot \ln x$

Differentiate

$\frac{f'(x)}{f(x)} = \cos x \cdot \ln x + \frac{\sin x}{x}$

$\Rightarrow f'(x) = f(x) \cdot \left(\cos x \cdot \ln x + \frac{\sin x}{x}\right) = x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right)$

o——x——x——o