Lecture 19

In this lecture we are going to find the derivative of $f(x) = e^x$ and $f(x) = a^x$, i.e., exponential functions.

Recall that we defined e as the unique number such that the graph of $f(x) = e^x$ has the tangent line with slope 1 at $x = 0$.

Now since slope of the tangent line at $x=0$ is precisely $f'(0)$

$\Rightarrow$ for $f(x) = e^x$, we have $f'(0) = 1$. So

$$1 = f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{e^h - 1}{h}$$

Thus we get the important fact

$$\boxed{\lim_{h\to 0} \frac{e^h - 1}{h} = 1} \quad \text{— } (1)$$

Let us use ① to find the derivative of $f(x) = e^x$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} \frac{e^x . e^h - e^x}{h}$$

$$= \lim_{h \to 0} e^x \underbrace{\left(\frac{e^h - 1}{h}\right)}_{=1 \text{ from } ①}$$

$$= e^x$$

So $\boxed{\frac{d}{dx} e^x = e^x}$

i.e., the derivative of e^x is e^x itself.

<u>Ques</u> Find the derivative of the following functions.

① $f(x) = 2e^x$

<u>Sol</u> $f'(x) = \frac{d}{dx}(2e^x) = 2e^x$

② $f(x) = xe^x$

<u>Sol</u> We use the product rule.

$f'(x) = \frac{d}{dx}(x) \cdot e^x + x \cdot \frac{d}{dx}(e^x)$

$= 1 \cdot e^x + x \cdot e^x = e^x(1+x).$

③ $f(x) = e^{\sqrt{x}}$

<u>Sol</u> Here we'll have to use the chain rule.

If $g(x) = e^x$ and $h(x) = \sqrt{x}$, then

$f(x) = e^{\sqrt{x}} = g(h(x))$

$\Rightarrow f'(x) = g'(h(x)) \cdot h'(x) = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$

④ $f(x) = e^{3x^2+5}$

Again, using the chain rule, we get

$f'(x) = e^{3x^2+5} \cdot (6x).$

What is the derivative of $f(x) = a^x$?

We note that $a = e^{\ln a} \Rightarrow a^x = e^{x \ln a}$

Thus, we can use the chain rule to get

$$\frac{d}{dx}(a^x) = e^{x\ln a}\cdot \ln a = a^x \ln a$$

so,

$$\boxed{\frac{d}{dx}(a^x) = a^x \ln a}$$

Remark :- You cannot apply the "Power Rule" here as ie a^x, the variable x is in the exponent

so $$\frac{d}{dx}(a^x) \neq x a^{x-1}$$

Ques. Find the derivative of the following functions.

1. $f(x) = 3\cdot 2^x$

$f'(x) = 3\cdot 2^x \ln 2$

2. $f(x) = x + 5^x$

$f'(x) = \frac{d}{dx}(x) + \frac{d}{dx}(5^x) = 1 + 5^x \ln 5$

3. $f(x) = 2^x \cdot e^x$

We use the product rule to get

$$f'(x) = \frac{d}{dx}(2^x) \cdot e^x + 2^x \cdot \frac{d}{dx}(e^x)$$

$$= 2^x \ln 2\, e^x + 2^x e^x = 2^x e^x (1 + \ln 2)$$

④ $$f(x) = \frac{x^2}{3^x + 1}$$

We'll use the quotient rule.

$$f'(x) = \frac{\frac{d}{dx}(x^2) \cdot (3^x + 1) - \frac{d}{dx}(3^x + 1) \cdot x^2}{(3^x + 1)^2}$$

$$= \frac{2x \cdot (3^x + 1) - 3^x \ln 3 \cdot x^2}{(3^x + 1)^2}$$

⑤ $$f(x) = 2^{5x^2 + 3}$$

We'll use the chain rule as $f(x) = g(h(x))$

with $g(x) = 2^x$ and $h(x) = 5x^2 + 3$

$$\Rightarrow \quad f'(x) = g'(h(x)) \cdot h'(x)$$

$$= 2^{5x^2+3} \cdot \ln 2 \cdot (10x).$$

<u>Ques</u> find the equation of the tangent line to $f(x) = x2^x$ at $x=0$.

<u>Soln</u> If the equation of the tangent line is

$$y = mx+b$$

then $m = f'(0)$

$$f'(x) = 2^x + x \cdot 2^x \ln 2$$

$$\Rightarrow f'(0) = 2^0 + 0 \cdot 2^0 \cdot \ln 2 = 1$$

$$\Rightarrow y = x+b$$

To find b, note that $f(0) = 0 \cdot 2^0 = 0$ and $(0,0)$ lie on the line.

$\Rightarrow 0 = 0+b \Rightarrow b=0$. Thus the equation of the tangent line is

$$y = x.$$

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