Recall :-

Differentiation Techniques

1) $\frac{d}{dx}(\text{constant}) = 0$ 2) $\frac{d}{dx}(x^n) = nx^{n-1}$

3) $\frac{d}{dx}(\underset{\downarrow}{k}\cdot f(x)) = kf'(x)$ 4) $\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$

constant

$$\frac{d}{dx}(f(x)) = f'(x)$$

5) $\frac{d}{dx}(f(x)\cdot g(x)) = f'(x)\cdot g(x) + g'(x)\cdot f(x)$

6) $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)\cdot g(x) - g'(x)\cdot f(x)}{(g(x))^2}$

Que. $f(x) = \frac{(3x-1)\cdot 2x}{\sqrt{x}}$ Find $f'(x)$.

Sol^n Use the quotient rule.

$$f'(x) = \frac{\frac{d}{dx}\left((3x-1)\cdot 2x\right)\cdot\sqrt{x} - \frac{d}{dx}(\sqrt{x})\cdot\left((3x-1)\cdot 2x\right)}{(\sqrt{x})^2}$$

$$f'(x) = \frac{\left(\underbrace{\frac{d}{dx}(3x-1)\cdot 2x + \frac{d}{dx}(2x)\cdot(3x-1)}_{=\text{ product rule on } \frac{d}{dx}\left((3x-1)\cdot 2x\right)}\right)\cdot\sqrt{x} - \left(\frac{1}{2\sqrt{x}}\right)\cdot\left((3x-1)\cdot 2x\right)}{x}$$

$$= \frac{\left[(3)\cdot 2x + 2\cdot(3x-1)\right]\cdot\sqrt{x} - \left(\frac{(3x-1)\cdot\cancel{2}\cancel{x}\sqrt{\cancel{x}}}{\cancel{2}\sqrt{\cancel{x}}}\right)}{x}$$

$$= \frac{\left[6x + 6x - 2\right]\sqrt{x} - \left((3x-1)\cdot\sqrt{x}\right)}{x}$$

$$= \frac{\left[12x-2\right]\sqrt{x} - \left((3x-1)\cdot\sqrt{x}\right)}{x} = \frac{12x^{\frac{3}{2}} - 2\sqrt{x} - 3x^{\frac{3}{2}} + \sqrt{x}}{x}$$

$$= \frac{9x^{\frac{3}{2}} - \sqrt{x}}{x}$$

Answer

Ques. Find the equation of the tangent line to $f(x) = \frac{(3x-1)\cdot 2x}{\sqrt{x}}$ at $x = 1$.

Sol. Suppose the equation of the tangent line is $y = mx + b$.

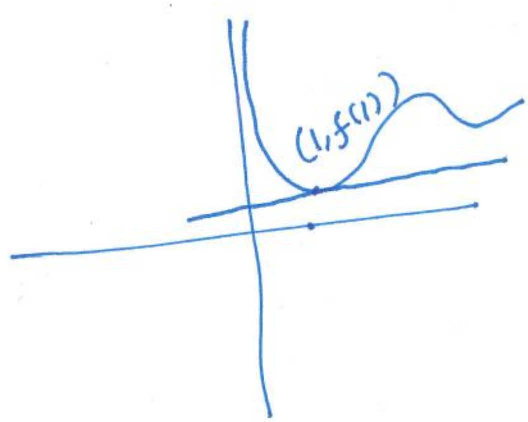
m = slope of the tangent line is just $f'(x)$ at $x = 1$.

$$f'(x) = \frac{9x^{\frac{3}{2}} - \sqrt{x}}{x}$$

$$\Rightarrow \quad m = f'(1) = \frac{9 \cdot 1^{\frac{3}{2}} - \sqrt{1}}{1} = 8$$

$$\therefore \quad y = 8x + b.$$

To find b, observe that the point $(1, f(1))$ lies on the tangent line.



$$f(1) = \frac{(3 \cdot 1 - 1) \cdot 2 \cdot 1}{\sqrt{1}}$$

$$= 4$$

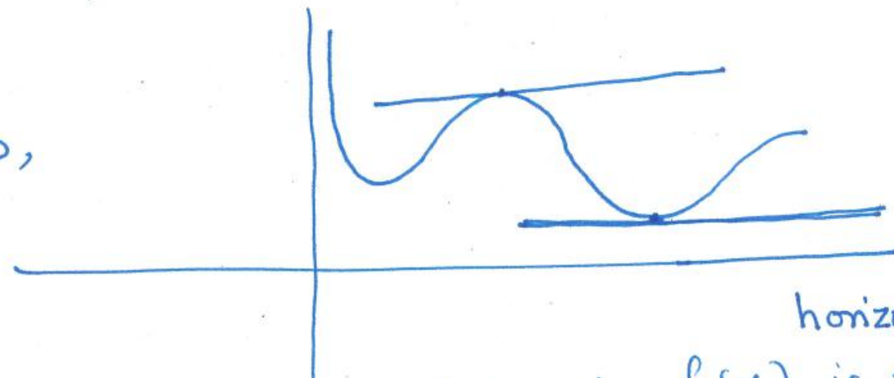
$$\Rightarrow \quad 4 = 8 \cdot 1 + b \quad \Rightarrow \quad b = -4$$

$\therefore$ equation of the line is

$$\boxed{y = 8x - 4}$$

<u>Ques</u> At what point x, does the graph of $f(x) = \dfrac{(3x-1)\cdot 2x}{\sqrt{x}}$, has a horizontal tangent line.

For horizontal lines, the slope $m = 0$.



$\therefore$ if at x, the tangent line to $f(x)$ is horizontal

$\Rightarrow$ $f'(x) = 0$, as $f'(x) =$ slope m of the tangent line.

$$\therefore f'(x) = \frac{9x^{\frac{3}{2}} - \sqrt{x}}{x} = 0 \Rightarrow 9x^{\frac{3}{2}} - \sqrt{x} = 0$$

$$\Rightarrow 9x^{3/2} = \sqrt{x}$$

$$\Rightarrow 9 = x^{-1}$$

$$\Rightarrow \boxed{x = \frac{1}{9}}$$

Chain Rule

Chain rule helps in finding the derivatives of composition of functions.

$f(x)$ and $g(x)$ then

$f(g(x))$ – replace all x's in the definition of $f(x)$ by $g(x)$

$g(f(x))$ – " ——— " ——— " of $g(x)$ by $f(x)$.

e.g. (i) $f(x) = x^2$, $g(x) = 3x-1$

$f(g(x)) = (3x-1)^2$, $g(f(x)) = 3x^2 - 1$

(ii) $f(x) = \sqrt{x}$, $g(x) = 3x^2 + 2x + 3$

$\Rightarrow f(g(x)) = \sqrt{3x^2 + 2x + 3}$

$g(f(x)) = 3(\sqrt{x})^2 + 2\sqrt{x} + 3 = 3x + 2\sqrt{x} + 3$

$$\boxed{\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x)}$$ – Chain rule.

differentiate $f(x)$ and compose it with $g(x)$

differentiate $g(x)$

multiply the above two.

$$\boxed{\frac{d}{dx}\left(g(f(x))\right) = g'(f(x)) \cdot f'(x)}$$

$$\underbrace{f(g(x)) = f \circ g\,(x)\ ,\ g(f(x)) = g \circ f\,(x)}_{\text{notation}}$$

e.g. ① $f(x) = x^2$, $g(x) = 3x-1$

$$\frac{d}{dx}\left(f \circ g(x)\right) = f'(g(x)) \cdot g'(x)$$

$$f'(x) = 2x \ , \quad g'(x) = 3$$

$$\Rightarrow \frac{d}{dx}\left(f(g(x))\right) = 2(3x-1) \cdot 3 = 6(3x-1)$$

$$= \boxed{18x-6}$$

$$\frac{d}{dx}\left(g(f(x))\right) = g'(f(x)) \cdot f'(x)$$

$$\Rightarrow \quad 3 \cdot 2x = \boxed{6x}$$

(no x was appearing in $g'(x)$)

(2) $f(x) = \sqrt{x}$, $g(x) = 3x^2 + 2x + 3$

$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x)$$

$$f'(x) = \frac{1}{2\sqrt{x}} , \quad g'(x) = 6x + 2$$

$$\Rightarrow \frac{d}{dx}\left(f(g(x))\right) = \frac{1}{2\sqrt{3x^2+2x+3}} \cdot (6x+2)$$

$$= \frac{3x+1}{\sqrt{3x^2+2x+3}}$$

Answer

(3) $f(x) = \dfrac{2x+5}{x}$, $g(x) = \sqrt{x} + 2x$

$$\frac{d}{dx}\left(f(g(x)\right) = ?$$

$$f'(x) = \frac{\frac{d}{dx}(2x+5)\cdot x - \frac{d}{dx}(x)\cdot(2x+5)}{x^2}$$

(quotient rule)

$$= \frac{2x - (2x+5)}{x^2} = \frac{-5}{x^2}$$

$$g'(x) = \frac{1}{2\sqrt{x}} + 2$$

$$\therefore \frac{d}{dx}\left(f(g(x)\right) = f'(g(x))\cdot g'(x)$$

$$= \frac{-5}{(\sqrt{x}+2x)^2}\cdot\left(\frac{1}{2\sqrt{x}} + 2\right)$$

$$= \frac{-5\left(\frac{1+4\sqrt{x}}{2\sqrt{x}}\right)}{(x+4x^2+2x^{3/2})}$$ → simplify.

$$= \frac{-5(1+4\sqrt{x})}{2\sqrt{x}(x+4x^2+2x^{3/2})}$$

Answer

Sometimes it may happen that it is not obvious that we have to apply the chain rule but it is indeed the case.

e.g. Ques find $f'(x)$ where $f(x) = (2x+1)^{101}$.

Solⁿ notice that $f(x) = h \circ g(x)$

where $h(x) = x^{101}$

$g(x) = 2x+1$

Thus $f'(x) = h'(g(x)) \cdot g'(x)$

$h'(x) = \cancel{100}\ 101x^{100}$

$g'(x) = 2$

$\Rightarrow f'(x) = 101(2x+1)^{100} \cdot 2$

$= 202(2x+1)^{100}$

Answer