Lecture 17

In this lecture, we'll learn various techniques to find the integral of functions.

① The constant rule

$$\boxed{\text{If } f(x) = k \text{ is a constant function then } f'(x) = 0}$$

e.g. if $f(x) = 2 \implies f'(x) = 0$

$f(x) = 10{,}000 \implies f'(x) = 0$

$f(x) = \pi \implies f'(x) = 0$

The reason for this is that, say $f(x) = k$, then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{k-k}{h} = 0.$$

② The power rule

$$\boxed{\text{If } f(x) = x^n \text{ then } f'(x) = nx^{n-1}.}$$

here $n \in \mathbb{R}$.

We already saw examples of this in Lec. 16 where we calculated $f'(x)$ for $f(x) = x, x^2$ and x^3.

Thus, e.g. $f(x) = x^5 \Rightarrow f'(x) = 5x^4$

$$f(x) = x^e \Rightarrow f'(x) = ex^{e-1}$$

$$f(x) = \sqrt{x} \Rightarrow f(x) = x^{\frac{1}{2}} \Rightarrow f'(x) = \frac{1}{2} x^{\frac{1}{2}-1}$$

$$= \frac{1}{2} x^{-1/2}$$

$$= \frac{1}{2\sqrt{x}}$$

$$f(x) = \frac{1}{\sqrt{x}} \Rightarrow f(x) = x^{-1/2}$$

$$\Rightarrow f'(x) = -\frac{1}{2} x^{-\frac{1}{2}-1} = -\frac{1}{2} x^{-3/2} = \frac{-1}{2\sqrt{x^3}}$$

and so on.

Now that we know the power and the constant rule, we can combine them to get

③ The coefficient rule

$$\boxed{\text{The derivative of } k \cdot f(x) \text{ is } k \cdot f'(x)}$$

i.e., we just differentiate the function and ignore the coefficient.

e.g. i) If $f(x) = 7x^3 \Rightarrow f'(x) = 7.(3x^2) = 21x^2$

ii) $f(x) = \frac{6}{x} \Rightarrow f'(x) = 6\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-6}{x^2}$

iii) $f(x) = e x^{\pi} \Rightarrow f'(x) = e\frac{d}{dx}(x^{\pi})$

$= e.\pi x^{\pi-1}$

What if we have sum/difference, product or quotient of two functions?

④ The sum rule

If $f(x)$ and $g(x)$ are two functions then

$$\boxed{\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)}$$

e.g. i) $f(x) = 7x^3 + 21x \Rightarrow f'(x) = 21x^2 + 21$

ii) $f(x) = 2x^2 - 7x + 3 \Rightarrow f'(x) = 4x - 7$

iii) $f(x) = 3x^2 - x + \sqrt{x} \Rightarrow f'(x) = 6x - 1 + \frac{1}{2\sqrt{x}}$

(5) The Product rule

following the sum rule, it is tempting to guess that

$$\frac{d}{dx}\left(f(x) \cdot g(x)\right) = f'(x) \cdot g'(x). \quad \times$$

Unfortunately, this is not true.

$$\boxed{\frac{d}{dx}\left(f(x) \cdot g(x)\right) = f'(x) \cdot g(x) + g'(x) \cdot f(x)}$$

To see why this is, let's use the definition to calcula-
-te $\frac{d}{dx}\left(f(x) \cdot g(x)\right)$.

$$\frac{d}{dx}\left(f(x) \cdot g(x)\right) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x) \cdot g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x) \cdot g(x+h) + f(x) \cdot g(x+h) - f(x) \cdot g(x)}{h}$$

(a trick here is adding and substracting $f(x) \cdot g(x+h)$).

$$= \lim_{h \to 0} \frac{g(x+h)\cdot\left(f(x+h)-f(x)\right) + f(x)\left(g(x+h)-g(x)\right)}{h}$$

$$= \lim_{h \to 0} g(x+h) \underbrace{\frac{\left(f(x+h)-f(x)\right)}{h}}_{= f'(x)} + \lim_{h \to 0} f(x) \underbrace{\frac{\left(g(x+h)-g(x)\right)}{h}}_{= g'(x)}$$

$$= g(x)\cdot f'(x) + f(x)\cdot g'(x)$$

This is why we have the product rule as mentioned above.

e.g. i) $f(x) = (2x+1)(7x^2-3)$

$$\Rightarrow f'(x) = \left(\frac{d}{dx}(2x+1)\right)\cdot(7x^2-3) + \left(\frac{d}{dx}(7x^2-3)\right)\cdot(2x+1)$$

$$= 2\cdot(7x^2-3) + (14x)(2x+1)$$

$$= 14x^2 - 6 + 28x^2 + 14x = 42x^2 + 14x - 6.$$

ii) $f(x) = (\sqrt{x}+3)\cdot 2x^2$

$$\Rightarrow f'(x) = \frac{d}{dx}(\sqrt{x}+3)\cdot x^2 + \frac{d}{dx}(x^2)\cdot(\sqrt{x}+3)$$

$$= \frac{1}{2\sqrt{x}}\cdot x^2 + 2x(\sqrt{x}+3)$$

$$= \frac{1}{2} x^{\frac{3}{2}} + 2x\sqrt{x} + 6x.$$

6 The quotient rule

$$\boxed{\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{\left(g(x)\right)^2}}$$

Thus, we see that $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) \neq \frac{f'(x)}{g'(x)}$

e.g. 1) $f(x) = \frac{3x^2+2}{2x-1}$

$$\Rightarrow f'(x) = \frac{\frac{d}{dx}(3x^2+2)\cdot(2x-1) - (3x^2+2)\frac{d}{dx}(2x-1)}{(2x-1)^2}$$

$$= \frac{(6x)(2x-1) - (3x^2+2)\cdot 2}{(2x-1)^2}$$

$$= \frac{6x^2 - 6x - 2}{(2x-1)^2}$$