Lecture 16

Let's recall from the last lecture that for a function $f(x)$, the instantaneous rate of change at $x = a$ is

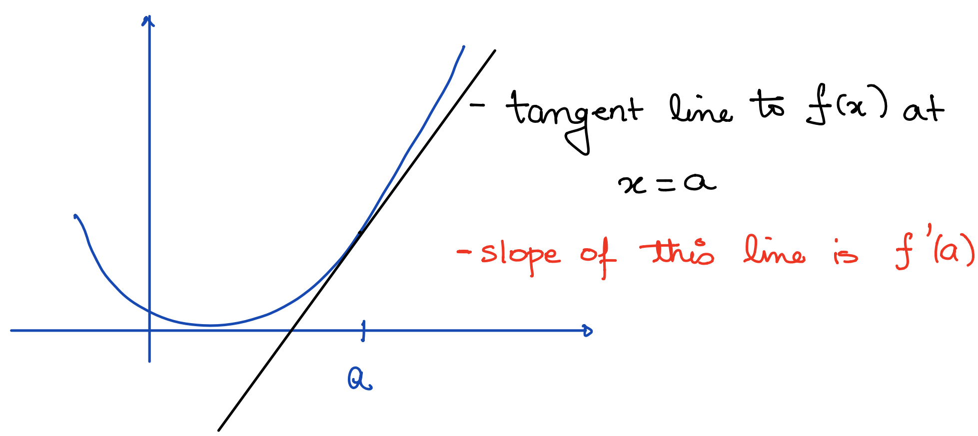
$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

This is also called the derivative of $f(x)$ at $x = a$.

We denote it by $f'(a)$.

It is also the slope of the tangent line at $x = a$.

The tangent line is the line that touches $f(x)$ at $x = a$ and at no other nearby points.



Given a function $f(x)$, we can think of its derivative as a function itself :-

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

Remark:- The derivative $f'(x)$ is also denoted as $\frac{d}{dx}(f(x))$.

Finding the derivative of a function using the definition

Let's see how to find the derivative of a function using the definition. If the question specifically says that then you must do it in this way. Later we will learn faster methods to find $f'(x)$.

Find $f'(x)$ for the following functions, using the definition.

1) $f(x) = x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h) - x}{h} = 1$$

Thus, $f'(x)$ is the constant function 1.

2) $f(x) = x^2$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{x^2+h^2+2hx-x^2}{h} = \lim_{h\to 0} \frac{h(2x+h)}{h} = \boxed{2x}$$

Thus, $f'(x)$ is the function $2x$.

3) $f(x) = x^3$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3-x^3}{h}$$

$$= \lim_{h\to 0} \frac{x^3+h^3+3x^2h+3xh^2-x^3}{h}$$

$$= \lim_{h\to 0} \frac{h(h^2+3x^2+3xh)}{h} = \boxed{3x^2}$$

[What do you think $f'(x)$ is for $f(x)=x^4$? Guess is $4x^3$]

5) $f(x) = \sqrt{x}$

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0} \frac{(\sqrt{x+h}-\sqrt{x})}{h} \cdot \frac{(\sqrt{x+h}+\sqrt{x})}{(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{x+h-x}{h\cdot(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \boxed{\frac{1}{2\sqrt{x}}}$$

E.g. find the equation of the tangent line to $f(x) = x^2+2x+3$ at $x=2$.

Sol^n :- Recall that the equation of a line is

$$y = mx + b$$

where m = slope which we saw is given by $f'(2)$

b = y-intercept

We calculate

$$m = f'(2) = \lim_{h\to 0} \frac{[(2+h)^2 + 2(2+h)+3] - [2^2 + 2.2+3]}{h}$$

$$= \lim_{h\to 0} \frac{(\cancel{4} + h^2 + 4h) + (\cancel{4} + 2h) + \cancel{3} - \cancel{4} - \cancel{4} - \cancel{3}}{h}$$

$$= \lim_{h\to 0} \frac{h(h+6)}{h} = 6$$

Thus, $y = 2x + b$

Also, the line is tangent to $f(x)$ at $x = 2 \Rightarrow$
$(2, f(2))$ is a point on the line and

$f(2) = 11 \quad \Rightarrow \quad 11 = 2.2 + b \Rightarrow b = 7.$

Thus, the equation of the tangent line is

$$\boxed{y = 6x + 7}$$

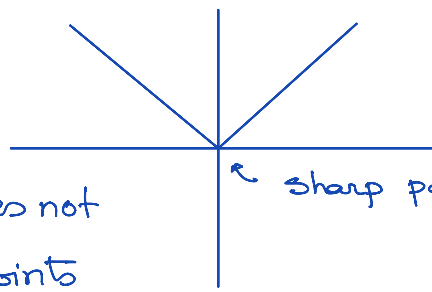
Let's calculate the derivative of $f(x) = |x|$ at $x = 0$.

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{|0+h| - |0|}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

This limit DNE as done in Lecture 13.

Thus $f'(x)$ for $|x|$ DNE at $x=0$!

This can be seen easily if we look at ~~the~~ graph of $|x|$.



sharp point at $x=0$.

The derivative does not exist at sharp points as there are several possible tangent lines.