Lecture 15

To understand the notion of derivatives, we have to understand rates of change.

Rates of Change

Suppose the distance travelled is recorded after every 30 minutes of the start of the travel :-

Time	0 hr	0.5 hr	1 hr	1.5 hrs	2 hrs	2.5 hrs	3 hr
Distance	0 Km	52 Km	100 Km	120 Km	175 Km	210 Km	250 Km

So average speed in 3 hrs is

$$\frac{\text{distance}}{\text{time}} = \frac{250}{3}\,\frac{\text{Km}}{\text{hr}} = 83.33 \text{ Km/hr}$$

What is the average speed in the first 1.5 hrs?

$$\frac{\text{distance in } 1^{st}\ 1.5 \text{ hrs}}{\text{time}} = \frac{120}{1.5} \text{ Km/hr} = 80 \text{ Km/hr}$$

What is the average speed in the last 1.5 hrs?

$$\frac{\text{Distance in last 1.5 hrs}}{\text{Time}} = \frac{250-120}{1.5}\ \frac{\text{km}}{\text{hr}} = \frac{130}{1.5}\ \text{km/hr}$$

$$\simeq 86.66\ \text{km/hr}$$

We can apply the same idea to functions.

Given a function $f(x)$, the average rate of change of $f(x)$ between $x=a$ and $x=b$ is

$$\boxed{\frac{f(b)-f(a)}{b-a}}$$

E.g. A runner's distance in metres over 60 seconds is given by

$$f(t) = t + 0.1t^2$$

1) What is the average speed of the runner in 60 seconds?

Sol Avg. speed $= \dfrac{f(60)-f(0)}{60-0} = \dfrac{60+0.1(60)^2}{60}$

$$= 7\ \text{m/s}$$

2) What is the runner's average speed over the last 10 seconds?

Sol^n Avg. speed $= \dfrac{f(60)-f(50)}{60-50}$

$$= \frac{[60+(0.1)(60)^2]-[50+0.1(50)^2]}{10}$$

$$= [6+36]-[5+25] = 12 \text{ m/s}$$

(3) Avg. speed in the last 1 second?

Sol. $\dfrac{f(60)-f(59)}{60-59} = \dfrac{[60+0.1(60)^2]-[59+0.1(59)^2]}{1}$

$$= [420]-[307.1] \text{ m/s}$$

$$= 12.9 \text{ m/s}$$

(4) The last millisecond? (0.001^{th} of a second)

Sol^n $\dfrac{f(60)-f(59.999)}{60-59.999} \simeq 12.99901 \text{ m/s}$

(5) What was the runner's instantaneous speed at $t=60$ s?

We guess it to be 13 m/s. To be sure, we need to calculate Limits!

The instantaneous rate of change of $f(x)$ at $x = a$ is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{(a+h) - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

So to calculate the runner's instantaneous speed at $t = 60\,s$ we calculate

$$\lim_{h \to 0} \frac{f(60+h) - f(60)}{h} = \lim_{h \to 0} \frac{[(60+h) + (0.1)(60+h)^2] - [60 + 0.1(60)^2]}{h}$$

$$= \lim_{h \to 0} \frac{[60+h + 0.1(3600 + h^2 + 120h)] - [420]}{h}$$

$$= \lim_{h \to 0} \frac{[13h + 0.1h^2]}{h} = 13\,m/s.$$

which is what we guessed.

e.g. find the instantaneous rate of change for $f(x) = x^2 + 2x$ at

(1) $x = 1$

Sol

$$\lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{[(1+h)^2+2(1+h)]-[1^2+2.1]}{h}$$

$$= \lim_{h\to 0} \frac{[1^2+h^2+2h+2+2h]-[3]}{h}$$

$$= \lim_{h\to 0} \frac{[4h+h^2]}{h} = 4$$

(2) $x=2$

Sol^n

$$\lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{[(2+h)^2+2(2+h)]-[2^2+2.2]}{h}$$

$$= \lim_{h\to 0} \frac{[4+h^2+4h+4+2h]-[4+4]}{h}$$

$$= \lim_{h\to 0} \frac{[h^2+6h]}{h} = 6$$

(3) at $x=a$.

Sol^n

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{[(a+h)^2+2(a+h)]-[a^2+2a]}{h}$$

$$= \lim_{h\to 0} \frac{[a^2+2ah+h^2+2a+2h-a^2-2a]}{h}$$

$$= \lim_{h\to 0} \frac{[2ah+2h]}{h} = 2a+2$$

We'll see that we have actually calculated the derivative of $f(x)$ at $x = a$!