# Calculus

3 main topics
- Limits
- Continuity
- Derivatives

## Limits

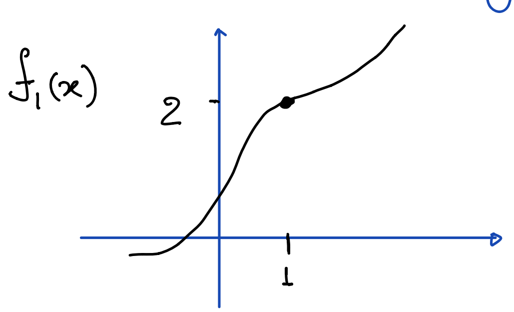
Uptill this point, we were more interested in finding the value of a function <u>at</u> a particular point,

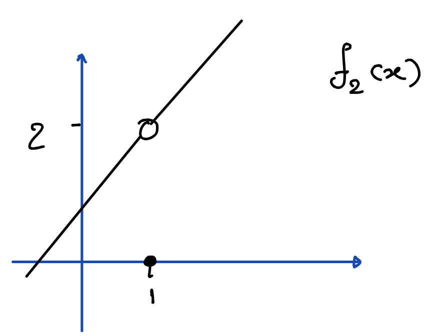
e.g. $f(x) = 2x+1$ at $x = 1$ gives

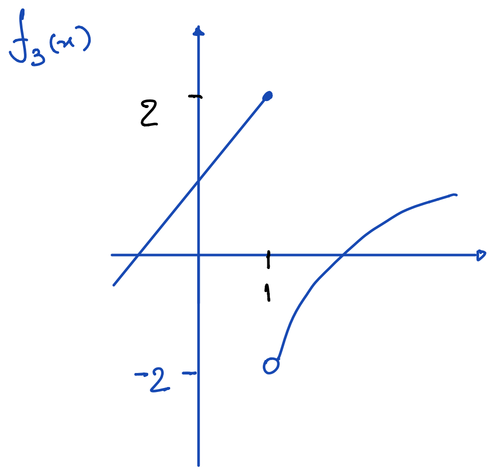
$$f(1) = 2\cdot 1+1 = 3$$

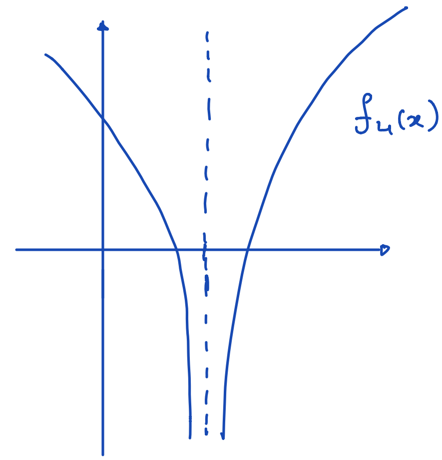
for limits, we will be interested in finding what happens to a function as $x$ gets infinitely close to some point $a$.

To understand the concept of limits, let's see the behaviour of the following graphs:






## Observations

Let's see what happens to the function as $x$ approaches 1.

$f_1(x)$ : - as $x$ approaches 1 from the left hand side $f_1(x)$ approaches the value 2

- as $x$ approaches 1 from the right hand side $f_1(x)$ approaches the value 2
- and $f_1(1) = 2$

$f_2(x)$ :- - as $x$ approaches 1 from the left hand side $f_2(x)$ approaches the value 2

- as $x$ approaches 1 from the right hand side $f_2(x)$ approaches the value 2
- $f_2(1) = 0$

$f_3(x)$ :- – as $x$ approaches 1 from the left hand side
$f_3(x)$ approaches the value 2.

– as $x$ approaches 1 from the right hand side,
$f_3(x)$ approaches the value $-2$

– $f_3(1) = 2$

$f_4(x)$ :- – as $x$ approaches 1 either from the left hand side or the right hand side, $f_4(x)$ goes to $-\infty$.

These examples motivate the following definition.

If $f(x)$ approaches a finite value L as $x$ approaches a or as $x$ gets infinitely close to a but not equal to a, we say

" the limit of $f(x)$ as $x$ approaches a is L"

and write
$$\boxed{\lim_{x \to a} f(x) = L}$$

**Note** :- 1) $L$ must be finite, i.e., $L \neq \pm\infty$.

2) $f(x)$ must approach the same value $L$, irrespective of whether $x$ approaches $a$ from the left hand side or the right hand side.

When $x$ approaches $a$ from the left then we say

$$\lim_{x \to a^-} f(x) = L \quad - \text{ Left hand limit (LHL)}$$

When $x$ approaches $a$ from the right then we says

$$\lim_{x \to a^+} f(x) = L \quad - \text{ Right hand limit (RHL)}$$

Thus for $\lim_{x \to a} f(x) = L$, we must have

$$\boxed{LHL = RHL = L}$$

3) We do not care about the value of $f(x)$ at $x = a$.

If $f(x)$ does not approach a finite value, or if $LHL \neq RHL$, then we say that

" the limit of $f(x)$ as $x$ approaches $a$ does not exist (DNE) "

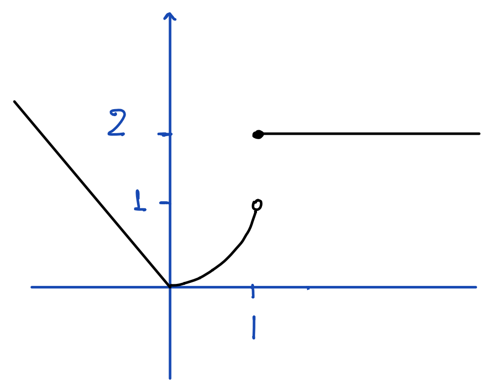
Thus for the previous graphs, we see

- $\lim_{x\to 1} f_1(x) = 2$
- $\lim_{x\to 1} f_2(x) = 2$
- $\lim_{x\to 1} f_3(x)$ DNE as LHL $= 2$ and RHL $= -2$
- $\lim_{x\to 1} f_4(x)$ DNE as even though LHL $=$ RHL, But LHL $= -\infty =$ RHL, not a finite value.

e.g. Suppose

$$f(x) = \begin{cases} -x & \text{if } x < 0 \\ x^2 & \text{if } 0 \le x < 1 \\ 2 & \text{if } x \ge 1 \end{cases}$$

graph




We see that

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (-x) = 0$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x^2) = 0$$

$$\Rightarrow \lim_{x\to 0} f(x) = 0$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 = 1$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2) = 2$$

$\Rightarrow \because$ LHL $\neq$ RHL so $\lim_{x\to 1} f(x)$ DNE.

Remember horizontal asymptotes? Now we can calculate them as well.

Horizontal asymptotes are calculated by calculating limit at $\pm\infty$, i.e., as $x \to +\infty$ or $x \to -\infty$.

The useful facts here are

$$\boxed{\lim_{x\to\infty} \frac{1}{x} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x} = 0}$$

## Limit Rules :-

Suppose $\lim_{x\to a} f(x) = F$ and $\lim_{x\to a} g(x) = G$

1. $\lim_{x\to a} \left(f(x) \pm g(x)\right) = F \pm G$

2. $\lim_{x\to a} \left(f(x) \cdot g(x)\right) = F \cdot G$

3. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{F}{G}$ (provided $G \neq 0$)

4. $\lim_{x\to a} \left(c \cdot f(x)\right) = c \cdot F$ where $c$ is a constant

5. $\lim_{x \to a} (f(x))^k = F^k$

6. $\lim_{x \to a} b^{f(x)} = b^F$

7. $\lim_{x \to a} \log_b f(x) = \log_b F$

8. $\lim_{x \to a} \sin(f(x)) = \sin(F)$

9. $\lim_{x \to a} \cos(f(x)) = \cos(F)$

Note :- When $f(x)$ is a polynomial then to calculate $\lim_{x \to a} f(x)$, we simply plug in $a$ in place of $x$.

In fact, this strategy might help many times.

E.g. Find the following limits.

a) $\lim_{x \to 1} x^2 + 3x + 4 = 1^2 + 3 \cdot 1 + 4 = 1 + 3 + 4 = 8$

b) $\lim_{x \to 2} \sqrt{x+2} = \sqrt{2+2} = 2$

c) $\lim_{x \to \frac{\pi}{3}} \sin\left(\frac{x}{2}\right) = \sin\left(\frac{\frac{\pi}{3}}{2}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

d) $\lim_{x \to 0} e^{x+1} - \ln(\sin(x)+1) = e^{0+1} - \ln(\sin(0)+1)$

$= e - \ln(1) = e$