Lecture 10

Growth and Decay problems

Many real life situations are modelled by the so called growth/decay problems. We can use exponentials and logarithms to solve them.

Let's look at the exponential growth/decay function.

Let $A(t)$ be the amount of a substance at time t

Then $$\boxed{A(t) = A_0 e^{kt}}$$

Here A_0 = initial amount of the substance or amount at time 0

t = time , k = constant

if $k > 0 \Rightarrow$ growth $\qquad k < 0 \Rightarrow$ decay

Typically, these problems will have some initial amount given (A_0) and some other data to find k and you will have to find the amount at some later time t.

E.g. Suppose a bacteria culture starts w/ 100 bacterias. After 3 hours it has 900 bacterias. How much will it have after 6 hours?

Sol^n Here initial amount, i.e., $A_0 = 100$

At $t = 3$, we have $A(3) = 900$

So, $900 = 100\, e^{3k}$. We want to find $A(6)$.

$\Rightarrow$ $9 = e^{3k}$ $\Rightarrow$ taking ln both sides

$\ln 9 = 3k \ln e$ $\Rightarrow$ $\ln 9 = 3k$ $\Rightarrow$ $k = \frac{\ln 9}{3}$

So our equation is $A(t) = 100\, e^{\left(\frac{\ln 9}{3}\right)t}$

Thus, $A(6) = 100\, e^{\frac{\ln 9}{3} \cdot 6} = 100\, e^{2 \ln 9}$

$= 100\, e^{\ln(9^2)} = 100\, e^{\ln 81}$

$= 8100$ (as $e^{\ln x} = x$).

Radioactive Decay [used for carbon dating]

Suppose a radioactive isotope has a half-life of 10 years. How much of this substance will be left after 30 years?

Sol^n Here we don't have A_0 explicitly, but initially the isotope would be 100%. Half-life means it will become half of its original value, i.e., 50%.

Thus in $A(t) = 100\, e^{kt}$, $A(10) = 50$

$$\Rightarrow \quad 50 = 100\, e^{10k} \quad \Rightarrow \quad \frac{1}{2} = e^{10k}$$

taking ln both sides gives $\ln(1/2) = \ln(e^{10k})$
$= 10k$

$$\Rightarrow \quad k = \frac{\ln(1/2)}{10}$$

So, $A(t) = 100\, e^{\frac{\ln(1/2)}{10} t}$. Need to find $A(30)$.

So, $$A(30) = 100\, e^{\frac{\ln(1/2)}{10} \cdot 30}$$

$$= 100\, e^{3\ln(1/2)} = 100\, e^{\ln\left(\frac{1}{8}\right)}$$

$$= \frac{100}{8} = \frac{25}{2} = 12.5\,\%$$

Chemical Dissolution

The amount of chemical that will dissolve in a solution can be modelled with the exponential growth function. The amount increases exponentially as temperature increases.

Say at $0^\circ C$, 10 g dissolves and at $10^\circ C$, 11 g dissolves.

Let $A(t)$ = amount that dissolves at temp. t

So $A(0) = 10 = A_0$

$A(10) = 11$

$\Rightarrow \quad 11 = 10\, e^{10k} \quad \Rightarrow \ln\left(\frac{11}{10}\right) = \ln(e^{10k}) = 10k$

$$\Rightarrow \quad k = \frac{\ln(11/10)}{10}$$

$$\therefore \quad A(t) = 10\, e^{\frac{\ln(11/10)}{10} \cdot t}$$

To find at what temperature 15 g dissolves, we need to find t.

$$\text{So,} \quad 15 = 10\, e^{\frac{\ln(11/10)}{10} t}$$

$$\Rightarrow \quad \frac{3}{2} = e^{\frac{1}{10}\ln(11/10) t}$$

$$\Rightarrow \quad \ln\left(3/2\right) = \frac{1}{10}\ln\left(11/10\right) t \quad \Rightarrow t = \frac{10\ln(3/2)}{\ln(11/10)}$$

$$\simeq 42.5^\circ C$$