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Recognizing *k*-Clique Extendible Orderings

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Abstract

We consider the complexity of recognizing k-clique-extendible graphs (k-C-E graphs) introduced by Spinrad (Efficient Graph Representations, AMS 2003), which are generalizations of comparability graphs. A graph is k-clique-extendible if there is an ordering of the vertices such that whenever two overlapping k-cliques A and B have $k-1$ common vertices, and these common vertices appear between the two vertices $a, b \in (A \setminus B) \cup (B \setminus A)$ in the ordering, there is an edge between a and b, implying that $A \cup B$ is a $(k+1)$-clique. Such an ordering is said to be a k-C-E ordering. These graphs arise in applications related to modelling preference relations. Recently, it has been shown that a maximum clique in such a graph can be found in $n^{O(k)}$ time [Hamburger et al. 2017] when the ordering is given. When k is 2, such graphs are precisely the well-known class of comparability graphs and when k is 3 they are called triangle-extendible graphs. It has been shown that triangle-extendible graphs appear as induced subgraphs of visibility graphs of simple polygons, and the complexity of recognizing them has been mentioned as an open problem in the literature. While comparability graphs (i.e. 2-C-E graphs) can be recognized in polynomial time, we show that recognizing k-C-E graphs is NP-hard for any fixed $k \geq 3$ and co-NP-hard when k is part of the input. While our NP-hardness reduction for $k \geq 4$ is from the betweenness problem, for $k = 3$, our reduction is an intricate one from the 3-colouring problem. We also show that the problems of determining whether a given ordering of the vertices of a graph is a k-C-E ordering, and that of finding a maximum clique in a k-C-E graph, given a k-C-E ordering, are hard for the parameterized complexity classes co-W[1] and W[1] respectively, when parameterized by k. However we show that the former is fixed-parameter tractable when parameterized by the treewidth of the graph. We also show that the dual parameterizations of all the problems that we study are fixed parameter tractable.

Keywords k-clique extendible orderings · k-clique extendible graphs · Comparability graphs · Hardness of recognition · Parameterized complexity

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Extended author information available on the last page of the article

1 Introduction and Motivation

An undirected graph is a comparability (or transitively orientable) graph if the edges can be oriented in a way that for any three vertices u, v, w whenever there is a (directed) edge from u to v and an edge from v to w, there is an edge from u to w. They are a well-studied class of graphs [5, 11] and they can be recognized in polynomial time [10]. Spinrad [14] generalized this class of graphs and introduced the notion of *k-clique-extendible orderings* (abbr. *k-C-E ordering*) on the vertices of a graph defined as follows. Here, and in the rest of the paper, we refer to a clique on k vertices in a graph as a *k-clique* in the graph.

Definition 1 *(k-C-E ordering, Spinrad* [14]*)* An ordering ϕ of the vertices of a graph $G = (V, E)$ is a k-clique-extendible ordering (or k-C-E ordering) of G if, whenever X and Y are two overlapping k-cliques such that $|X \cap Y| = k - 1$, $X \backslash Y = \{a\}$, $Y \backslash X = \{b\}$, and all the vertices in $X \cap Y$ occur between a and b in ϕ, we have $(a, b) \in E(G)$ and hence $X \cup Y$ is a $(k + 1)$-clique.

A graph G is said to be *k-clique-extendible* (*k-C-E* for short) if there exists a k-clique-extendible ordering ϕ of G. It can be observed that comparability graphs are exactly the 2-clique-extendible graphs. Spinrad [14] observed that 3-clique-extendible graphs, also called triangle-extendible graphs, arise in the visibility graphs of simple polygons and that a maximum clique can be found in polynomial time in such graphs if a 3-clique-extendible ordering is given. This result has been generalized to obtain an $n^{O(k)}$ algorithm for finding a maximum clique in k-C-E graphs (given with a k-C-E ordering) on n vertices [6]. Spinrad poses as an open problem the question of whether there is a polynomial time algorithm to recognise 3-C-E graphs (open problem 14.1 in [14]).

We believe that k-C-E graphs are natural generalizations of comparability graphs and our main contribution in this paper is a serious study of this class of graphs. We study two problems on k-C-E graphs: Recognition and verification. In the former, the goal is to recognize whether a given graph is a k-C-E graph (and give a witnessing k-C-E ordering), and in the latter, the goal is to verify whether a given ordering is a k-C-E ordering of the graph. We define these problems formally as follows.

FIND k-C-E ORDERING

Input: Graph G and an integer k
Question: Is G a k-C-E graph?

VERIFY k-C-E ORDERING

Input: Graph G, an integer k and an ordering ϕ of $V(G)$
Question: Is ϕ a k-C-E ordering of G?

Our results show that recognizing k-C-E graphs (the problem FIND k-C-E ORDERING) is NP-hard for any fixed $k \geq 3$ and also co-NP-hard when k is part of the input.

This solves the open problem regarding the complexity of recognizing 3-C-E graphs and we hope that our results will trigger further study of k-C-E graphs in general.

As for the problem Verify k-C-E Ordering, if a graph and an ordering of its vertices is given, then it is easy to get an $n^{O(k)}$ algorithm to determine whether it is a k-C-E ordering of the graph (see Sect. 4). We show that this problem is co-NP-complete and also complete for the parameterized complexity class co-W[1]. The reduction also implies that unless the Exponential Time Hypothesis fails, this problem does not have an $f(k)n^{o(k)}$ algorithm for any function f of k. However, we show that the problem is fixed-parameter tractable when parameterized by the treewidth of the graph, that is, there is an $f(tw)n^{O(1)}$ algorithm for the problem, where tw is the treewidth of the graph (see Sect. 2 for definitions).

We also look at the problem of finding a maximum clique in a k-C-E graph parameterized by k.

Clique in k-C-E Graph

Input: k-C-E graph G, a k-C-E ordering ϕ of G and an integer p
Parameter: k
Question: Does there exist a clique on at least p vertices in G?

Hamburger et al. [6] show that, given a k-C-E ordering ϕ of a k-C-E graph, one can compute the maximum clique in time $n^{O(k)}$. We show that this is most likely optimal, that is, there is no $f(k)n^{o(k)}$ algorithm for finding maximum clique in a k-C-E graph, even if given the k-C-E ordering, unless ETH fails.

Finally, we study the algorithmic complexity of the dual parameterizations of the recognition and verification problems. The problem of verifying whether a vertex ordering of a graph on n vertices is an $(n-k)$-C-E ordering and the problem of determining if an input graph on n vertices has an $(n-k)$-C-E ordering are both shown to be fixed parameter tractable. These problems are defined formally below.

Find $(n-k)$-C-E Ordering

Input: Graph G on n vertices and an integer k
Parameter: k
Question: Is G an $(n-k)$-C-E graph?

Verify $(n-k)$-C-E Ordering

Input: Graph G on n vertices, an integer k and an ordering ϕ of $V(G)$
Parameter: k
Question: Is ϕ an $(n-k)$-C-E ordering of G?

Organization of the paper. In the next section, we give the necessary notation and definitions. In Sect. 3, we prove some results about k-C-E graphs which are used in our reductions in later sections. In Sect. 4, we show that the problem of checking whether a given ordering is a k-C-E ordering is co-NP-complete and co-W[1]-complete. In this section, we also show that the problem is fixed-parameter tractable when parameterized by the treewidth of the graph. In Sect. 5 we show that the $n^{O(k)}$ algorithm for finding maximum clique in a k-C-E graph [6] is likely optimal. Section 6 gives our main NP-hardness reductions for the problem of recognizing k-C-E graphs. We give two reductions, one for $k = 3$ and another for $k \geq 4$. In Sect. 7, we show that the dual parameterizations of the recognition and verification problems are both fixed parameter tractable. We list some open problems in Sect. 8.

2 Preliminaries

Definition 2 *(Fixed-Parameter Tractability)* A parameterized problem (or a language) $L \subseteq \Sigma^* \times \mathbb{N}$ is said to be fixed-parameter tractable (FPT) if there exists an algorithm $\mathcal{B}$, a constant c and a computable function $f : \mathbb{N} \times \mathbb{N}$ such that given any $(I, k) \in \Sigma^* \times \mathbb{N}$, $\mathcal{B}$ runs in at most $f(k) \cdot |I|^c$ time and decides correctly whether $(I, k) \in L$ or not. Here f is a function only of k, and c is a constant independent of k. We call algorithm $\mathcal{B}$ as fixed-parameter algorithm, and we also denote a runtime like $f(k)|I|^c$, a FPT runtime. FPT also denotes the class of fixed-parameter tractable problems. Here $|I|$ is the size of the input and k is the parameter.

There is also an intractability (hardness) theory in parameterized complexity captured by parameterized reductions defined below.

Definition 3 *(Parameterized Reduction)* There is a parameterized reduction from a parameterized problem P_1 to a parameterized problem P_2, if every instance (x, k) of P_1 can be transformed in FPT time to an equivalent instance (x', k') where k' is just a function of k.

There is a hierarchy of complexity classes $FPT \subseteq W[1] \subseteq W[2] \subseteq \cdots \subseteq XP$. It is believed that the containments are strict and there are canonical complete problems under parameterized reductions. The Clique problem that asks whether a given undirected graph has a clique on k vertices is a canonical $W[1]$-complete problem, where k is the parameter. Parameterized problems that have a parameterized reduction to the Clique problem form the class $W[1]$. We refer readers to the textbook [3] for further discussions on parameterized complexity.

Definition 4 *(Tree-decomposition and treewidth* [13]*)* A *tree decomposition* of a graph G is a pair $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$, where T is a tree whose every node t is assigned a vertex subset $X_t \subseteq V(G)$, called a bag, such that the following three conditions hold : (*i*) $\bigcup_{t \in V(T)} X_t = V(G)$. (*ii*) For every $uv \in E(G)$, there exists a node t of T such that bag X_t contains both u and v. (*iii*) For every $u \in V(G)$, the set $T_u = \{t \in V(T) \mid u \in X_t\}$ induces a subtree of T. The width of tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ equals $\max_{t \in V(T)} \{|X_t| - 1\}$. The treewidth of a graph G, denoted by $tw(G)$, is the minimum possible width of a tree decomposition of G.

The following conjecture, known as the Exponential Time Hypothesis, is used to provide lower bounds for hard problems.

Exponential Time Hypothesis (ETH) [7]: There is a positive real s such that 3-CNF-SAT cannot be solved in time 2^{sn} time where n is the number of variables.

In particular, ETH states that 3-CNF-SAT cannot be solved in time $2^{o(n)}$. See also [8] for a survey of various lower bound results using ETH.

All graphs considered in this paper are undirected and simple. Given a graph G, by $V(G)$ we denote the set of vertices in the graph and by $E(G)$ we denote the set of edges in the graph. Let G be a graph. For a subset of vertices $S \subseteq V(G)$, we define $G[S]$ as the induced subgraph of G having vertex set S.

Given a linear order ϕ of a set A, we write $a <_\phi b$ to mean that a and b are two elements of A such that a occurs before b in ϕ. Also, we write $\phi = (a_1, a_2, \dots, a_n)$ to mean that $A = \{a_1, a_2, \dots, a_n\}$ and $a_1 <_\phi a_2 <_\phi \cdots <_\phi a_n$. We say that a vertex b comes between vertices a and c in ϕ if $a <_\phi b <_\phi c$ or $c <_\phi b <_\phi a$. By ϕ^{-1} we denote the reverse of ϕ, that is, $a <_{\phi^{-1}} b$ if and only if $b <_\phi a$.

Given an ordering ϕ of a set V and a set $S \subseteq V$, we define $\phi|_S$ to be the ordering of the elements of S in the order in which they occur in ϕ. Further, we say that $a, b \in S$ are the endpoints of S if a is the first element of $\phi|_S$ and b is the last element of $\phi|_S$. Given two disjoint sets A and B, and orderings $\phi_1 = (a_1, a_2, \dots, a_n)$ of the set A and $\phi_2 = (b_1, b_2, \dots, b_m)$ of the set B, we define $\phi_1 \mathbin{+\!\!\!+} \phi_2$ to be the ordering $(a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_m)$ of the set $A \cup B$, that is, $\mathbin{+\!\!\!+}$ is the concatenation operator on orderings. We will abuse notation to allow sets of orderings to be used with the concatenation operator: if A and B are disjoint sets, Φ a set of orderings of A, and Ψ a set of orderings of B, then we denote by $\Phi \mathbin{+\!\!\!+} \Psi$ the set of orderings $\{\phi \mathbin{+\!\!\!+} \psi : \phi \in \Phi, \psi \in \Psi\}$ of $A \cup B$. Further, for a set A, we shall abuse notation so that A also denotes the set of all orderings of A. We do not distinguish between a set that contains just one ordering and the ordering itself; i.e. we denote $\{\phi\}$ by just ϕ. Thus, if ϕ is an ordering of a set A, and B is a set disjoint from A, then $\phi \mathbin{+\!\!\!+} B$ denotes the set of all orderings of $A \cup B$ in which elements of A appear in the order given by ϕ as the first $|A|$ elements, followed by the elements of B in any order.

A clique in a graph is a set of vertices that are pairwise adjacent in the graph. An independent set is a set of vertices that are pairwise non-adjacent. Given subsets $S, A, B \subseteq V(G)$, we say that S separates A and B if there is no path from A to B in $G[V(G) \setminus S]$. For a pair u, v of nonadjacent vertices of a graph, by identifying u with v, we mean adding the edges (u, w) for all $w \in N(v) \setminus N(u)$ and then deleting v.

We denote by K_n^- the graph obtained by removing an edge from the complete graph K_n on n vertices. Given an ordering ϕ of the vertices of a graph G, we say that an induced subgraph H of G is an *ordered* K_t^- in ϕ if $\phi|_{V(H)} = (h_1, h_2, \dots, h_t)$ and $E(H) = \{(h_i, h_j) \mid 1 \leq i < j \leq t\} \setminus \{(h_1, h_t)\}$. It follows that an ordering of the vertices of a graph is a k-C-E ordering if and only if it contains no ordered K_{k+1}^-.

3 Basic Results

We start with the following observations which are used throughout the paper.

Observation 1 An ordering $\phi = \{v_1, v_2, \dots, v_n\}$ is a k-C-E ordering, if and only if its reverse ordering, $\phi^{-1} = \{v_n, v_{n-1}, \dots, v_1\}$ is also a k-C-E ordering.

Observation 2 Given a graph G and an induced subgraph H of G, if an ordering ϕ is a k-C-E ordering of G, then $\phi|_{V(H)}$ is a k-C-E ordering of H. Thus every induced subgraph of a k-clique-extendible graph is also k-clique-extendible.

Observation 3 If G is a k-colourable graph with colour classes $V_1, \dots, V_k$, then any ordering $\phi \in V_1 \mathbin{+\!\!\!+} V_2 \mathbin{+\!\!\!+} \cdots \mathbin{+\!\!\!+} V_k$ is a k-C-E ordering of G. Thus, every k-colourable graph is k-clique-extendible.

It is not difficult to see that any k-clique-extendible ordering of a graph is also a $(k+1)$-clique-extendible ordering. Thus, every k-clique-extendible graph is also a $(k+1)$-clique-extendible graph. Note that every graph on n vertices is trivially n-clique-extendible. So the notion of k-clique-extendibility gives rise to a hierarchy of graph classes starting with comparability graphs and ending with the entire set of graphs. This motivates the use of k as a graph parameter.

We prove a lemma that will help us construct a k-C-E ordering of a graph from k-C-E orderings of its subgraphs.

Lemma 1 *For a graph G, let $V_1, V_2 \subseteq V(G)$ and let σ_1, σ_2 be k-C-E orderings of $G[V_1]$ and $G[V_2]$ respectively for any $k \geq 2$, such that the following hold*

1. $V_1 \cup V_2 = V(G)$
2. $V_1 \cap V_2$ *separates G into components $V_1 \backslash V_2$ and $V_2 \backslash V_1$*
3. $\sigma_1|_{V_1 \cap V_2} = \sigma_2|_{V_1 \cap V_2}$
4. *if C is a $(k-1)$-clique in $V_1 \cap V_2$ and u, v are the endpoints of C in σ_1, then every vertex $a \in V_1 \backslash V_2$ that is adjacent to all of C satisfies $u <_{\sigma_1} a <_{\sigma_1} v$*

Then G has a k-C-E ordering ϕ such that $\phi|_{V_1} = \sigma_1$ and $\phi|_{V_2} = \sigma_2$.

Proof Let $\sigma_1|_{V_1 \cap V_2} = \sigma_2|_{V_1 \cap V_2} = (s_1, s_2, \dots, s_p)$. Let α_i be the induced ordering between s_i and s_{i+1} in σ_1, so we can rewrite σ_1 as

$$\sigma_1 = \alpha_0 \mathbin{+\!\!\!+} (s_1) \mathbin{+\!\!\!+} \alpha_1 \mathbin{+\!\!\!+} (s_2) \mathbin{+\!\!\!+} \alpha_2 \mathbin{+\!\!\!+} \cdots \mathbin{+\!\!\!+} \alpha_{p-1} \mathbin{+\!\!\!+} (s_p) \mathbin{+\!\!\!+} \alpha_p$$

Similarly, let β_i be the induced ordering between s_i and s_{i+1} in σ_2 so that

$$\sigma_2 = \beta_0 \mathbin{+\!\!\!+} (s_1) \mathbin{+\!\!\!+} \beta_1 \mathbin{+\!\!\!+} (s_2) \mathbin{+\!\!\!+} \beta_2 \mathbin{+\!\!\!+} \cdots \mathbin{+\!\!\!+} \beta_{p-1} \mathbin{+\!\!\!+} (s_p) \mathbin{+\!\!\!+} \beta_p$$

Consider the following ordering ϕ of $V_1 \cup V_2$.

$$\phi = \alpha_0 \mathbin{+\!\!\!+} \beta_0 \mathbin{+\!\!\!+} (s_1) \mathbin{+\!\!\!+} \alpha_1 \mathbin{+\!\!\!+} \beta_1 \mathbin{+\!\!\!+} (s_2) \mathbin{+\!\!\!+} \alpha_2 \mathbin{+\!\!\!+} \beta_2 \mathbin{+\!\!\!+} \cdots \mathbin{+\!\!\!+} \alpha_{p-1} \mathbin{+\!\!\!+} \beta_{p-1} \mathbin{+\!\!\!+} (s_p) \mathbin{+\!\!\!+} \alpha_p \mathbin{+\!\!\!+} \beta_p$$

That is, we 'interleave' each α_i and β_i between the corresponding s_i and s_{i+1}. The ordering ϕ is constructed such that it preserves the internal ordering of σ_1 in V_1 and σ_2 in V_2, that is, $\phi|_{V_1} = \sigma_1$ and $\phi|_{V_2} = \sigma_2$ and thus also $\phi|_{V_1 \cap V_2} = \sigma_1|_{V_1 \cap V_2} = \sigma_2|_{V_1 \cap V_2}$.

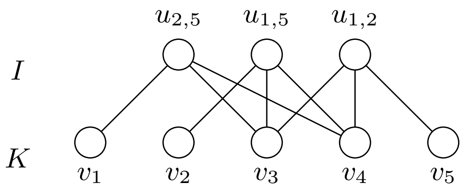

Fig. 1 Diagram depicting F_3. Edges in the clique are not shown, and only 3 of the $u_{i,j}$ vertices are shown to avoid visual clutter.

We will prove that ϕ is a k-C-E ordering of G. Suppose not. Then there exists a set $Q \subseteq V(G)$ that forms an ordered K^-_{k+1} in ϕ. Let a, b be the endpoints of Q in ϕ. It can't be the case that $Q \subseteq V_1$, otherwise since $\phi|_{V_1} = \sigma_1$, Q would be an ordered K^-_{k+1} in σ_1, contradicting the fact that σ_1 is a k-C-E ordering of $G[V_1]$. Similarly, it can't be the case that $Q \subseteq V_2$. So, $Q \cap (V_1 \backslash V_2) \neq \emptyset$ and $Q \cap (V_2 \backslash V_1) \neq \emptyset$. As $V_1 \cap V_2$ separates V_1 and V_2, no vertex in $V_1 \backslash V_2$ is adjacent to any vertex $V_2 \backslash V_1$. Since the only two vertices in Q that do not have an edge between them are a and b, we can assume without loss of generality that $a \in V_1 \backslash V_2$ and $b \in V_2 \backslash V_1$, and we further get that $Q \backslash \{a, b\} \subseteq V_1 \cap V_2$. Since $Q \backslash \{a, b\}$ is a $(k-1)$-clique and a is adjacent to all the vertices of $Q \backslash \{a, b\}$, by the last condition in the lemma, it must be the case that a lies between the two endpoints of $Q \backslash \{a, b\}$ in ϕ, contradicting the fact that a is an endpoint of Q in ϕ. □

Forbidden subgraph. We construct a forbidden subgraph for the class of k-clique-extendible graphs which is used to build gadgets in our NP-hardness reductions.

For a positive integer k, let $K = \{v_1, v_2, \dots, v_{2k-1}\}$ be a $(2k-1)$-clique. For every pair of vertices v_i and v_j in K, add a vertex $u_{i,j}$ such that $u_{i,j}$ is adjacent to every vertex in K except v_i and v_j. Let $I = \{u_{i,j} \mid i, j \in [2k-1], i < j\}$ be the set of all such $u_{i,j}$ for every pair of vertices in K. Let F_k be the graph thus obtained having vertex set $K \cup I$. See Fig. 1 for an example that demonstrates the adjacencies between I and K when $k = 3$.

Lemma 2 *F_k is not k-clique-extendible.*

Proof Suppose not. Let ϕ be a k-C-E ordering of F_k. By the symmetry in F_k between the vertices of K, we can assume without loss of generality that $\phi|_K = (v_1, v_2, \dots, v_{2k-1})$. If the vertex $u_{1,2k-1}$ comes after v_k in ϕ, then the vertices in $\{v_1, v_2, \dots, v_k, u_{1,2k-1}\}$ form an ordered K^-_{k+1}. On the other hand, if $u_{1,2k-1}$ comes before v_k in ϕ, then the vertices in $\{u_{1,2k-1}, v_k, v_{k+1}, \dots, v_{2k-2}, v_{2k-1}\}$ form an ordered K^-_{k+1}. In both cases, we get an ordered K^-_{k+1}, so ϕ cannot be a k-C-E ordering, which contradicts our assumption. □

4 Verifying a *k*-C-E Ordering

In this section, we prove that even verifying whether an ordering is a k-clique-extendible ordering is hard (assuming k is considered as part of the input, rather than a constant).

VERIFY k-C-E ORDERING

Input: Graph G, integer k and an ordering ϕ of $V(G)$
Question: Is ϕ a k-C-E ordering of G?

VERIFY k-C-E ORDERING has a simple $n^{O(k)}$ algorithm as one can enumerate all $\binom{n}{k+1}$ subgraphs isomorphic to K^-_{k+1}, and check if any of them are ordered with respect to the ordering. We prove that the problem is CO-W[1]-complete and CO-NP-complete by a reduction from and to the CLIQUE problem, and that the problem also cannot have a $f(k)n^{o(k)}$ algorithm assuming ETH (see Sect. 2 for a definition). The reduction maps the YES instances of VERIFY k-C-E ORDERING to the NO instances of CLIQUE and vice-versa, hence showing that VERIFY k-C-E ORDERING is CO-NP-complete.

Theorem 1 *VERIFY k-C-E ORDERING is CO-W[1]-complete, CO-NP-complete and there is no $f(k)n^{o(k)}$ algorithm for it unless ETH fails.*

Proof We will prove hardness first by giving a reduction from CLIQUE. In the CLIQUE Problem, we are given a graph G and a positive integer k and asked to check whether there exists a k-clique in G.

Given G, let $\phi = (v_1, v_2, \dots, v_n)$ be an arbitrary ordering of its vertices. We construct $G' = (V', E')$, where $V' = V \cup \{a, b\}$ and $E' = E \cup \{(a, v), (b, v) \mid v \in V\}$. We then ask whether the ordering $\sigma = (a, v_1, v_2, \dots, v_n, b)$ is a $(k+1)$-C-E ordering of G'. We claim that G has a k-clique if and only if σ is not a $(k+1)$-C-E ordering of G'.

Suppose G contains a k-clique C. Then the vertices in $\{a, b\} \cup C$ form an ordered K^-_{k+2} in σ. Thus we conclude that σ is not a $(k+1)$-C-E ordering. Conversely, if G does not have a k-clique, then G' cannot have a $(k+1)$-clique, so any ordering is trivially a $(k+1)$-C-E ordering of G'.

The above reduction proves that the problem is CO-W[1]-hard and CO-NP-hard. Since CLIQUE cannot be solved in time $f(k)n^{o(k)}$ unless ETH fails [3, 8], the above reduction implies that VERIFY k-C-E ORDERING cannot be solved in time $f(k)n^{o(k)}$ unless ETH fails. It remains to show that the problem is in CO-W[1] and in CO-NP. For this, we give a reduction to the CLIQUE problem.

Given an ordering σ of vertices of G, we do the following. For every pair of non-adjacent vertices (u, v), let $V_{u,v}$ be the set of common neighbours of both u and v, that appear between u and v in the ordering. That is, $w \in V_{u,v}$ if and only if $(w, u), (w, v) \in E(G)$ and $u <_\sigma w <_\sigma v$ or $v <_\sigma w <_\sigma u$. Let $G_{u,v}$ be the induced subgraph $G[V_{u,v}]$. Let $E(\overline{G})$ denote the pairs of vertices in G which are non-adjacent. We define $G' = \dot\bigcup_{(u,v)\in E(\overline{G})} G_{u,v}$. That is, G' is the disjoint union of $G[V_{u,v}]$ for all pairs

(u, v) that are non-adjacent. We claim that G' has a $(k-1)$-clique if and ony if σ is not a k-C-E ordering.

Suppose G' has a $(k-1)$-clique C. The clique must be in a connected component of G', say $G_{u,v}$. Then the vertices u and v are non-adjacent in G, and by construction of $G_{u,v}$, every vertex $w \in C$ is such that w lies between u and v in σ and w is a neighbour to both u and v. Thus $\{u, v\} \cup C$ forms an ordered K^-_{k+1} in σ and hence σ is not a k-C-E ordering. Conversely, suppose there is an ordered K^-_{k+1} in σ. Let Q be the vertices of the ordered K^-_{k+1} and let u and v be its endpoints in σ. Then $C = Q \backslash \{u, v\}$ forms a $(k-1)$-clique such that every vertex $w \in C$ is a neighbour to both u and v and lies between u and v in σ. Therefore C forms a $(k-1)$-clique in $G_{u,v}$. □

If all the k-cliques in a graph can be enumerated in time $f(k)n^{O(1)}$, for some function f, then we can verify if an ordering is a k-C-E ordering in $g(k)n^{O(1)}$ time, for some function g, by checking every pair of such cliques to see if they form an ordered K^-_{k+1}. We show that a similar situation happens if G has bounded treewidth and so the verification problem becomes easy.

Lemma 3 (see for example [3]) *For any clique K in G, there exists a vertex $v \in V(T)$ such that all the vertices of K appear in the bag B_v corresponding to the vertex v in the tree decomposition.*

Lemma 4 ([1]) *There exists an algorithm, that given an n-vertex graph G and an integer t, runs in time $2^{O(t)} \cdot n$ and either constructs a tree decomposition of G of width at most $5t + 4$ and $n^{O(1)}$ bags, or concludes that the treewidth of G is greater than t.*

Theorem 2 *Given an ordering of the vertices of a graph G on n vertices, we can verify whether it is a k-C-E ordering of G in time $tw^{O(tw)}n^{O(1)}$, where tw is the treewidth of G.*

Proof Due to Lemma 3, if $tw < k - 1$, G will be a trivial k-C-E graph as it cannot contain any k-cliques. So, for the problem to remain non-trivial, we assume that $k - 1 \leq tw$.

We use Lemma 4 to obtain a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ of G of width $w \leq 5tw + 4$ that has $n^{O(1)}$ bags. We can verify whether an ordering is a k-clique-extendible ordering in $O(w^k n^{O(1)})$ time as follows. From Lemma 3, every $(k-1)$-clique must appear in one bag. As the bag sizes are bounded by $w + 1$, and the number of bags is $n^{O(1)}$, we can enumerate all $(k-1)$-cliques within a bag in $w^k n^{O(1)}$ time. Now, for every such clique K and for every pair of vertices (u, v) in G that are non-adjacent, we check whether $K \subseteq N(u) \cap N(v)$ and the set of vertices of K appear between u and v in the ordering. If they do for at least one such clique K and vertex pair (u, v), we output "no", otherwise we output "yes".

The algorithm takes $O(w^k n^{O(1)})$ time. Since k is upper bounded by $tw + 1$ and $w = O(tw)$, this runtime is $tw^{O(tw)} n^{O(1)}$. □

5 Hardness of Finding Clique

In this section, we look at the following problem of finding a maximum clique in a k-C-E graph, parameterized by k.

Clique in k-C-E Graph

Input: k-C-E graph G, a k-C-E ordering ϕ of G and an integer p
Parameter: k
Question: Does there exist a clique on at least p vertices in G?

There exists an $n^{O(k)}$ algorithm for finding a maximum clique in a k-C-E graph [6] when a k-C-E ordering is given. We will prove that this is most likely optimal, that is, we prove that unless ETH fails, there is no $f(k)n^{o(k)}$ algorithm for finding a maximum clique in a k-C-E graph *even if the ordering is given*. We do this by giving a parameter preserving reduction from the Multicolored Clique problem, defined below.

Multicoloured Clique

Input: Graph G, a partition $V_1, \dots, V_k$ of $V(G)$
Question: Does there exist a k-clique C in G such that $|C \cap V_i| = 1$ for each $i \in [k]$?

Multicoloured Clique is W[1]-hard and cannot be solved in time $f(k)n^{o(k)}$ unless ETH fails [3]. Given an instance $G, V_1, \dots, V_k$ of Multicoloured Clique, we will first remove all edges that lie within each partition V_i. Hence the graph G is now k-colourable with colour classes $V_1, \dots, V_k$. Any k-colourable graph is also a k-C-E graph by Observation 3. Thus we can give the graph G and an ordering $\phi \in V_1 \mathbin{+\!\!\!+} V_2 \mathbin{+\!\!\!+} \cdots \mathbin{+\!\!\!+} V_k$ as input to any algorithm for the problem Clique in k -C-E graph, setting $p = k$, to find if there is a clique on at least k vertices in G. If the the algorithm returns "yes", then we output "yes", otherwise we output "no". Since there is also an obvious reduction from Clique in k-C-E Graph to Clique, we have the following theorem.

Theorem 3 *The problem Clique in k-C-E graph is NP-complete, W[1]-complete, and cannot be solved in time $f(k)n^{o(k)}$ unless ETH fails.*

More specifically, our reduction actually shows that even the more restricted problem of deciding if there is a clique on at least k vertices in a k-C-E graph (even if given a k-C-E is ordering of it) is NP-complete, W[1]-complete and cannot be done in time $f(k)n^{o(k)}$ unless ETH fails.

6 Finding a k-C-E Ordering

In this section, we consider the following problem and prove the main result of the paper.

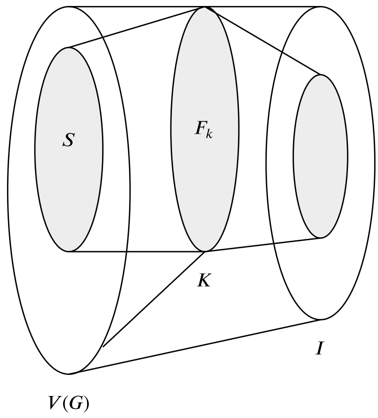

Fig. 2 Diagram depicting the reduction for Theorem 4. The shaded region shows F_k as an induced subgraph.

FIND k-C-E ORDERING

Input: Graph G, integer k
Question: Is G a k-C-E graph?

Note that this is possibly a harder problem than VERIFY k-C-E ORDERING, but still Theorem 1 doesn't immediately imply even CO-W[1]-hardness for this problem, as one may be able to determine whether G has a k-C-E ordering without even verifying an ordering. Our main result in this section is to show that FIND k-C-E ORDERING is NP-hard for each $k \geq 3$. First we will show that FIND k-C-E ORDERING is CO-W[1]-hard and CO-NP-hard. This result rules out algorithms running in time $f(k)n^{o(k)}$ assuming ETH (where as the NP-hardness for fixed k rules out even $n^{f(k)}$ algorithms assuming P≠NP).

Theorem 4 *FIND k-C-E ORDERING is CO-W[1]-hard and CO-NP-hard.*

Proof We will reduce from the CLIQUE problem. Given an integer k and a graph G in which we wish to find a k-clique, we construct another graph G' such that G' contains the forbidden subgraph F_k (defined in Sect. 2) if and only if it has a clique on k vertices.

Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let $K = \{v_{n+1}, v_{n+2}, \ldots, v_{n+k-1}\}$ be a $(k-1)$-clique such that each vertex in K is connected to every vertex in $V(G)$ and let $I = \{u_{i,j} \mid i,j \in [n+k-1], i < j\}$ such that $u_{i,j}$ is adjacent to all vertices in $\{v_1, v_2, \ldots, v_{n+k-1}\}$ except for v_i and v_j. Let G' be the graph where $V(G') = V(G) \cup I \cup K$.

Claim *G has a k-clique if and only if G′ does not have a k-C-E ordering.*

Proof See Fig. 2 for a figure depicting the constructed graph G'. Suppose S is a k-clique in G. Then $S \cup K \cup I$ will have an induced subgraph isomorphic to F_k such that the $(2k-1)$-clique of F_k is $K \cup S$ and the independent set of F_k is a subset of I. By Lemma 2, F_k does not have a k-C-E ordering and hence, by Observation 2, G' does not have a k-C-E ordering.

Conversely if G does not have a k-clique then any arbitrary ordering of $V(G)$ will be a k-C-E ordering of G. We will argue that any ordering $\pi \in V(G) +\!\!\!+ I +\!\!\!+ K$ is a k-C-E ordering of G'.

It is enough to prove that there does not exist an ordered K_{k+1}^- in π. For contradiction, suppose $Q \subseteq V(G')$ forms an ordered K_{k+1}^- in π. Let a, b be the endpoints of Q in π so that $(a, b) \notin E(G')$ and $a <_\pi b$. Let $A = Q \setminus \{b\}$ and $B = Q \setminus \{a\}$. Note that A and B are two k-cliques and $A \cup B = Q$. Since I is an independent set (and $k \geq 2$), Q contains at most one vertex from I. Therefore, if $a \in I \cup K$, then $B \subseteq K$, which is a contradiction as $|B| > |K|$. Similarly, if $b \in V(G) \cup I$, then $A \subseteq V(G)$, which is a contradiction as A is then a k-clique in G. We thus have $a \in V(G)$ and $b \in K$. But then $(a, b) \in E(G')$, which is a contradiction. Therefore, we conclude that there cannot be an ordered K_{k+1}^- in π. □

The reduction maps the Yes instances of Clique to the No instances of Find k-C-E Ordering and vice-versa. Hence Find k-C-E Ordering is co-W[1]-hard and co-NP-hard. □

6.1 NP-hardness for $k \geq 4$

We now prove the NP-hardness of Find k-C-E Ordering by a reduction from Betweenness defined below. The reduction strategy works for all $k \geq 4$ but not for $k = 3$ and so we give a different reduction for $k = 3$ in the next section.

Betweenness

Input: Universe U of size n, and a set of triples $\mathcal{T} = \{t_1, \ldots, t_m\}$ where each $t_i = (a_i, b_i, c_i)$ is an ordered triple of elements in U

Question: Does there exist an ordering ϕ of U such that either $a_i <_\phi b_i <_\phi c_i$ or $c_i <_\phi b_i <_\phi a_i$ for each triple $(a_i, b_i, c_i) \in \mathcal{T}$?

Betweenness is NP-hard [12]. To prove our reduction, we will require a gadget that takes as input a graph G and 3 vertices $x, y, z \in V(G)$ and converts them to a modified graph G' in such a way that either $x <_\phi y <_\phi z$ or $z <_\phi y <_\phi x$ for any k-C-E ordering ϕ of G'. Moreover, if ϕ is a k-C-E ordering of G such that $x <_\phi y <_\phi z$ or $z <_\phi y <_\phi x$ then ϕ is also a k-C-E ordering of G' (more formally, our construction will have $V(G) \subseteq V(G')$, and we will prove that G' has a k-C-E ordering ψ such that $\psi|_{V(G)} = \phi$). Thus the gadget 'prunes' out the orderings of the graph where y does not lie between x and z in the ordering. The k-C-E orderings of G' are exactly the k-C-E orderings ϕ of G where either $x <_\phi y <_\phi z$ or $z <_\phi y <_\phi x$. Thus to construct the reduction, we will start with a graph where all $n!$ orderings are valid k-C-E orderings, and apply the gadget for each $(a_i, b_i, c_i) \in \mathcal{T}$. After applying the gadgets, we will have pruned out all the 'bad' orderings and we will remain with exactly the set of orderings in which b_i lies between a_i and c_i for each $i \in [m]$. To describe the construction of the gadget, first we need to define an auxiliary graph Γ_k.

Definition of the auxiliary graph. Recall the graph F_k, defined in Sect. 2 on the vertex set $K \cup I$ where $K = \{v_1, v_2, \dots, v_{2k-1}\}$ induces a clique on $2k-1$ vertices, and every vertex in I is indexed by a pair of vertices of K to which the vertex is not adjacent. Pick arbitrary vertices v_1 and v_2 of K and let $u_{1,2}$ be the vertex of I that is adjacent to every vertex of K except v_1 and v_2. Define $\Gamma_k = F_k \setminus \{u_{1,2}\}$. Note that Γ_k has $O(k^2)$ many vertices.

Lemma 5 *In any k-C-E ordering ϕ of Γ_k, v_1 and v_2 are the endpoints of K. Furthermore, there exists a k-C-E ordering ϕ of Γ_k such that v_1 is the first element in ϕ and v_2 is the last.*

Proof Suppose that ϕ is a k-C-E ordering of Γ_k and suppose for contradiction that v_i and v_j are the endpoints of K where $\{i,j\} \neq \{1,2\}$. Then there exists $u_{i,j} \in I$ that is adjacent to every vertex in K except v_i and v_j. Let $\phi|_K = (v_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(2k-1)})$ where σ is a permutation of $\{1, 2, \dots, 2k-1\}$ such that $\sigma(1) = i$ and $\sigma(2k-1) = j$.

If the vertex $u_{i,j}$ comes after $v_{\sigma(k)}$ in ϕ, then the vertices in $\{v_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(k)}, u_{i,j}\}$ form an ordered K^-_{k+1} in ϕ. On the other hand, if the vertex $u_{i,j}$ comes before $v_{\sigma(k)}$ in ϕ, then the vertices in $\{u_{i,j}, v_{\sigma(k)}, v_{\sigma(k+1)}, \dots, v_{\sigma(2k-2)}, v_{\sigma(2k-1)}\}$ form an ordered K^-_{k+1} in ϕ. In both cases, we get a contradiction to ϕ being a k-C-E ordering. Therefore every k-C-E ordering ϕ of Γ_k is such that v_1 and v_2 are the endpoints of K.

Now we will prove the existence of an ordering ϕ of Γ_k such that v_1 is the first element of ϕ and v_2 is the last. Let I_1 be the set of all vertices in I that are not adjacent to v_1 and let $I_2 = I \setminus I_1$. Note that, since we have removed the vertex $u_{1,2}$ from F_k to get Γ_k, all vertices in I_1 are adjacent to v_2 and all vertices in I_2 are adjacent to v_1.

Consider an ordering $\phi \in (v_1, v_3, v_4, v_5, \dots, v_k) \mathbin{+\!\!\!+} I_1 \mathbin{+\!\!\!+} (v_{k+1}) \mathbin{+\!\!\!+} I_2 \mathbin{+\!\!\!+} (v_{k+2}, v_{k+3}, \dots, v_{2k-2}, v_{2k-1}, v_2)$. We claim that ϕ is a k-C-E ordering of Γ_k. Observe that v_1 is the first element of ϕ and v_2 is the last, thus we will be done once we prove the claim.

Suppose that ϕ is not a k-C-E ordering of Γ_k, then there exists $Q \subseteq V(\Gamma_k)$ that induces an ordered K^-_{k+1} in ϕ. Let a, b be the endpoints of Q in ϕ so that $(a, b) \notin E(\Gamma_k)$ and $a <_\phi b$. Let $A = Q \setminus \{b\}$ and $B = Q \setminus \{a\}$. Note that A and B are k-cliques and $A \cup B = Q$.

Note that at most one vertex from I can be contained in Q because otherwise $a, b \in I$, which implies that there is at most one vertex (which is v_{k+1}) between a and b in Q, contradicting the fact that $k \geq 3$. If $a \in I$, then by the above observation, we have $B \subseteq \{v_{k+1}, v_{k+2}, \dots, v_{2k-1}, v_2\}$. Since $|B| = k$, we have $B = \{v_{k+1}, v_{k+2}, \dots, v_{2k-1}, v_2\}$, which implies that $a \in I_1$ and $b = v_2$. By our earlier observation, all vertices in I_1 are adjacent to v_2, which contradicts the fact that $ab \notin E(\Gamma_k)$. We thus have that $a \notin I$. By a symmetric argument, we get that $b \notin I$. Then $a, b \in K$, which again contradicts the fact that $ab \notin E(\Gamma_k)$. □

The Gadget. We will use Γ_k as a gadget to constrict the set of orderings a graph can have. Pick an arbitrary vertex $v_3 \in K$ such that $v_3 \neq v_1, v_2$. Given a graph G,

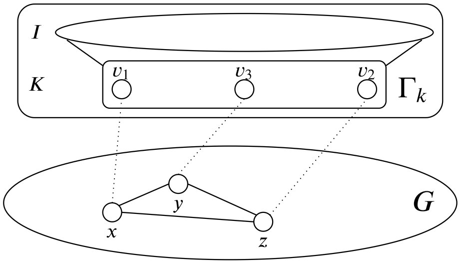

Fig. 3 The construction of the gadget. Dotted lines indicate vertices identified to each other.

applying the gadget on a triplet of vertices $x, y, z \in V(G)$ involves taking the disjoint union of G and Γ_k and identifying the vertices x with v_1, y with v_3 and z with v_2 (See Fig. 3). For technical reasons, we will only be applying the gadget on vertices x, y, z that induce a clique in G. Since Γ_k has $O(k^2)$ many vertices, the gadget will add $O(k^2)$ vertices to G, keeping it well within a polynomial factor. We use notation $G' = \mathcal{C}_k(G, x, y, z)$ to denote "G' is obtained by applying the gadget on G on vertices x, y, z". The valid k-C-E orderings of G' should exactly be the k-C-E orderings of G where y comes between x and z. The following lemmas give us exactly that.

Lemma 6 *Let G be a graph and let $x, y, z \in V(G)$ be vertices of G. In any k-C-E ordering ϕ of $G' = \mathcal{C}_k(G, x, y, z)$, y comes between x and z.*

Proof Γ_k is a subgraph of G'. By Observation 2, any k-C-E ordering of G' must induce a k-C-E ordering of Γ_k within it. Furthermore, v_3 comes between v_1 and v_2 in any k-C-E ordering of Γ_k by Lemma 5 and since x, y, z are identified with v_1, v_3, v_2 respectively, it follows that y is between x and z in any k-C-E ordering of G'. □

Lemma 7 *Let $k \geq 4$ and let G be a graph that has a k-C-E ordering ψ such that y comes between x and z for some three vertices $x, y, z \in V(G)$ that form a 3-clique in G, then $G' = \mathcal{C}_k(G, x, y, z)$ has a k-C-E ordering ϕ such that $\phi|_{V(G)} = \psi$.*

Proof By Lemma 5, Γ_k has a k-C-E ordering σ where v_1 and v_2 are the first and last elements in σ respectively. Since v_1 is identified with x and v_2 with z, we have that $x <_\sigma a <_\sigma z$ for each $a \in V(\Gamma_k) \setminus \{x, z\}$. We wish to use Lemma 1 on ψ and σ to obtain a k-C-E ordering of G'. Observe that $V(G) \cup V(\Gamma_k) = V(G')$ and $V(G) \cap V(\Gamma_k) = \{x, y, z\}$ separates $V(G)$ and $V(\Gamma_k)$, thus the first and second conditions in Lemma 1 hold. Since x, y, z form a clique in both G and in Γ_k, we have that $G'[V(G)]$ is isomorphic to G and $G'[V(\Gamma_k)]$ is isomorphic to Γ_k. Thus ψ is a k-C-E ordering of $G'[V(G)]$ and σ is a k-C-E ordering of $G'[V(\Gamma_k)]$. Without loss of generality, we can assume by Observation 1, that $x <_\psi y <_\psi z$ in G. It also holds that, $x <_\sigma y <_\sigma z$ since y is identified with $v_3 \in V(\Gamma_k) \setminus \{x, z\}$. Therefore $\psi|_{\{x,y,z\}} = \sigma|_{\{x,y,z\}}$, and the third condition also holds. Let a be a vertex in $V(\Gamma_k) \setminus \{x, y, z\}$. Suppose there exists a clique $C \subseteq V(G) \cap V(\Gamma_k) = \{x, y, z\}$ on $k - 1$ vertices that are all adjacent to a. Since $k \geq 4$, it follows that $|C| \geq 3$, and thus $x, z \in C$ are the endpoints of C in σ.

By the property of σ, we have $x <_\sigma a <_\sigma z$. Thus all four conditions for Lemma 1 are satisfied and the lemma follows. □

The Reduction. We are now ready to prove that the problem of checking whether a graph has a k-C-E ordering is NP-hard for each $k \geq 4$.

Theorem 5 *Find k-C-E Ordering is NP-hard for each* $k \geq 4$.

Proof We will reduce from Betweenness. Let $\mathcal{I} = (U, \mathcal{T})$ be the input Betweenness instance. We want to construct a graph G' such that G' has a k-C-E ordering if and only if the Betweenness instance is satisfiable. We will construct a graph with vertex set equal to the universe U and apply the gadget for every triple (a_i, b_i, c_i) in $\mathcal{T}$. We do this iteratively, that is, we first define G_0 to be the complete graph on vertex set U and then construct $G_i = \mathcal{C}_k(G_{i-1}, a_i, b_i, c_i)$ for each $i \in [m]$ (where m is the number of triples in $\mathcal{T}$). The final graph G' is equal to G_m. There are m many calls to the gadget and each gadget adds $O(k^2)$ vertices to G'. So the final size of G' is $O(n + mk^2)$ (where n is the size of U), which is polynomial in n and m.

Claim *G' has a k-C-E ordering if and only if $\mathcal{I}$ is a Yes instance.*

Proof **(⇒)** Recall that the graph G' is constructed iteratively as follows. We first define G_0 to be the complete graph on the vertex set U and then $G_i = \mathcal{C}_k(G_{i-1}, a_i, b_i, c_i)$ for $i \in [m]$. The final graph G' is G_m. Suppose G' has a k-C-E ordering ϕ. Let $\psi = \phi|_U$. We claim that ψ is a valid betweenness ordering for the instance $\mathcal{I}$. Let $(a_i, b_i, c_i) \in \mathcal{T}$ be a triple. Consider the subgraph G_i of G'. By Lemma 6, b_i comes between a_i and c_i in any k-C-E ordering of G_i. Since G_i is a subgraph of G, by Observation 2, b_i must come between a_i and c_i in any k-C-E ordering of G' as well. This holds for every triple $(a_i, b_i, c_i) \in \mathcal{T}$, thus ψ is a valid betweenness ordering for the instance $\mathcal{I}$.

(⇐) Let ψ be a valid betweenness ordering of U. We will prove that there exists a k-C-E ordering ϕ of G' such that $\phi|_U = \psi$. Clearly, G_0 has such an ordering. We proceed by induction on $i = 1, \ldots, m$. Suppose, by the induction hypothesis, that G_i has a k-C-E ordering ϕ_i such that $\phi_i|_U = \psi$. Note that b_i comes between a_i and c_i in ψ (and thus also in ϕ_i) as it is a valid betweenness ordering. By Lemma 7, G_{i+1} has a k-C-E ordering ϕ_{i+1} such that $\phi_{i+1}|_{V(G_i)} = \phi_i$ and thus also $\phi_{i+1}|_U = \phi_i|_U = \psi$. Thus the induction step follows. □

The theorem follows from the above claim. □

The above reduction shows that Find k-C-E Ordering is NP-hard. From Theorem 4, Find k-C-E Ordering is also co-NP-hard. Thus it is unlikely that the problem is in NP or in co-NP. Moreover, it is easy to verify that the problem lies in Σ_2^P, as one can simply guess the ordering ϕ and use a co-NP machine (Theorem 1) to check whether ϕ is a k-C-E ordering. Thus it is an open question whether Find k-C-E Ordering is Σ_2^P-complete.

Remark 1 It is important to note that the problem is *not* co-NP-hard when k is a fixed constant as opposed to it being given as a input. When k is fixed, the k-C-E ordering itself is an NP certificate for the problem, as given an ordering it is easy to check whether it is a k-C-E ordering for constant k. Thus, when k is constant, the problem is NP-complete. Indeed, the proof of co-NP-hardness in Theorem 1 assumes that k is given as an input.

Remark 2 The reduction in Theorem 5 does not work for $k = 3$ due to technicalities that arise in order to satisfy the fourth condition of Lemma 1, due to which we require that $|V(G) \cap V(\Gamma_k)| \leq k - 1$ (see Proof of Lemma 7). Since $|V(G) \cap V(\Gamma_k)| = |\{x, y, z\}| = 3$ in the gadgets we construct, this forces k to be at least 4. We give a separate proof for NP-hardness of $k = 3$ in the following section that uses some different ideas.

6.2 NP-hardness for $k = 3$

In this section, we prove that the problem of finding a 3-C-E ordering is NP-hard. We will reduce from the 3-Colouring problem. Given a graph G and an ordering ϕ of $V(G)$, we say that three edges $(u, v), (w, x), (y, z) \in E(G)$ form a *disjoint triple* in ϕ if $u <_\phi v \leq_\phi w <_\phi x \leq_\phi y <_\phi z$. Here $x \leq_\phi y$ means that either $x = y$ or $x <_\phi y$.

Observation 4 Let G be a 3-colourable graph and let C_1, C_2, C_3 be a partition of $V(G)$ into three independent sets. Then any ordering $\phi \in C_1 \mathbin{+\!\!+} C_2 \mathbin{+\!\!+} C_3$ contains no disjoint triple.

Proof Suppose that $(u, v), (w, x), (y, z) \in E(G)$ form a disjoint triple in ϕ, where $u <_\phi v \leq_\phi w <_\phi x \leq_\phi y <_\phi z$. For a vertex $a \in V(G)$, let $c(a)$ denote the integer $i \in \{1, 2, 3\}$ such that $a \in C_i$. Since $u <_\phi v$, $(u, v) \in E(G)$, and $\phi \in C_1 \mathbin{+\!\!+} C_2 \mathbin{+\!\!+} C_3$, it must be the case that $c(v) \geq c(u) + 1$. Since $v \leq_\phi w$, we then get $c(w) \geq c(v) \geq c(u) + 1$. Similarly, as $(w, x) \in E(G)$, $c(x) \geq c(w) + 1 \geq c(u) + 2$, and further, as $x \leq y$, we get $c(y) \geq c(u) + 2$. Continuing in this fashion, since $(y, z) \in E(G)$, we get $c(z) \geq c(u) + 3$. But as $c(u) \geq 1$, we now have $c(z) > 3$, which is a contradiction. □

Observation 5 Let G be any graph. If there is an ordering ϕ of $V(G)$ that contains no disjoint triple, then G is 3-colourable.

Proof We orient the edges of G using ϕ. The edge (u, v) is oriented from u to v if $u <_\phi v$. Since there is no disjoint triple in ϕ, there cannot be a directed path of length 3 in the oriented graph. By the Gallai-Roy-Vitaver Theorem (see Chapter 5 of [15]), we then have that G is 3-colourable. □

It follows from Observations 4 and 5 that a graph G is 3-colourable if and only if there is an ordering of its vertex set containing no disjoint triple.

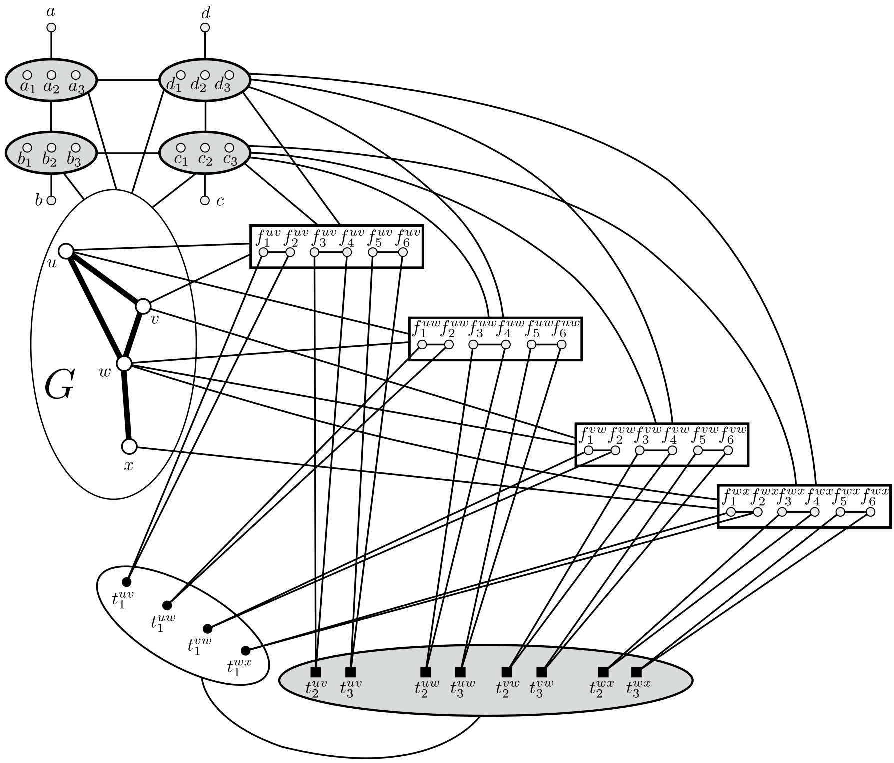

Fig. 4 The construction of G' from G. The vertices inside each shaded block form a clique. An edge between a vertex u and a block means that u is adjacent to every vertex in the block, and an edge between two blocks means that every vertex in one block is adjacent to every vertex in the other block. Note that an edge between vertices a and b is denoted as ab instead of (a, b) to reduce clutter.

Another observation is that in any 3-C-E ordering ϕ of G, for any pair of non-adjacent vertices $u, v \in V(G)$, the vertices that are adjacent to both u and v and lie between u and v in ϕ must be an independent set in G. Indeed, if there is an edge (a, b) such that $u <_\phi a <_\phi b <_\phi v$ and a, b are adjacent to both u and v, then $G[\{u, a, b, v\}]$ is an ordered K_4^- in ϕ. This suggests a reduction from 3-Colouring. The idea is that, associated to every edge $e = (u, v) \in E(G)$, we will add a vertex t_1^e, and a pair of adjacent vertices t_2^e and t_3^e. We will add edges so that the t_2 vertices and t_3 vertices together form a clique and the t_1 vertices form an independent set. We also add edges between all t_2, t_3 vertices and t_1 vertices. We will add a gadget to ensure that t_1^e, t_2^e, t_3^e all lie between u and v in any 3-C-E ordering of G'.

If G is not 3-colourable, then for any ordering ϕ of $V(G')$, there will be a disjoint triple in $\phi|_{V(G)}$. If the disjoint triple is formed by the edges (u, v), (w, x), (y, z) of G, where $u <_\phi v \leq_\phi w <_\phi x \leq_\phi y <_\phi z$, then the vertices $t_1^{(u,v)}, t_2^{(w,x)}, t_3^{(w,x)}, t_1^{(y,z)}$ form an ordered K_4^- in ϕ, and hence there can be no 3-C-E ordering of G'. On the other hand, our construction makes sure that if G is a 3-colourable graph, then there exists a 3-C-E ordering for G'. We now describe the reduction in detail.

The Construction. Given a graph G, we construct a supergraph G' as explained below (also see Fig. 4). For subsets $A, B \subseteq V(G)$, by "join A and B", we mean that we add all possible edges between vertices in A and vertices in B. To construct the vertex set of G', we take the vertex set of G and add the following.

1. Add 4 sets of vertices $A = \{a, a_1, a_2, a_3\}$, $B = \{b, b_1, b_2, b_3\}$, $C = \{c, c_1, c_2, c_3\}$ and $D = \{d, d_1, d_2, d_3\}$
2. Add the sets of vertices $F = \{f_i^e \mid e \in E(G), i \in \{1, 2, \dots, 6\}\}$ and $T = \{t_i^e \mid e \in E(G), i \in \{1, 2, 3\}\}$

To construct the edge set of G', we take the edge set of G and add the following.

1. Add edges to make A, B, C and D into cliques on 4 vertices each.
2. Add edges to make $t_i^e, f_{2i-1}^e, f_{2i}^e$ into a clique, for each edge $e \in E(G)$ and $i \in [3]$
3. Join $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$
4. Join $\{b_1, b_2, b_3\}$ and $\{c_1, c_2, c_3\}$
5. Join $\{c_1, c_2, c_3\}$ and $\{d_1, d_2, d_3\}$
6. Join $\{d_1, d_2, d_3\}$ and $\{a_1, a_2, a_3\}$
7. Join $\{a_1, a_2, a_3, b_1, b_2, b_3\}$ and $V(G)$
8. Join $\{c_1, c_2, c_3, d_1, d_2, d_3\}$ and $V(G) \cup F$
9. Add edges $(f_i^{(u,v)}, u)$ and $(f_i^{(u,v)}, v)$, for each $(u, v) \in E(G)$ and $i \in \{1, \dots, 6\}$
10. Add edges to make $\bigcup_{e \in E(G)} \{t_2^e, t_3^e\}$ into a clique
11. Join $\bigcup_{e \in E(G)} t_1^e$ and $\bigcup_{e \in E(G)} \{t_2^e, t_3^e\}$

Lemma 8 *If G' has a 3-C-E ordering then G is 3-colourable.*

Proof Suppose that ϕ is a 3-C-E ordering of G'. By Observation 5, we only need to show that there is no disjoint triple in the ordering $\phi|_{V(G)}$. We can assume without loss of generality that there exist distinct $i, j \in \{1, 2, 3\}$ such that in the ordering ϕ, we have $a_i <_\phi a_j <_\phi a$ (reversing the ordering ϕ if necessary; recall Observation 1). If there is a vertex $w \in V(G) \cup \{b_1, b_2, b_3, d_1, d_2, d_3\}$ such that $w <_\phi a_i$, then w, a_i, a_j, a form an ordered K_4^- in ϕ, which contradicts the fact that ϕ is a 3-C-E ordering. Therefore, we can assume without loss of generality that the vertex a_1 occurs before every vertex of $V(G) \cup \{b_1, b_2, b_3, d_1, d_2, d_3\}$ in the ordering ϕ. This also means that if there exist distinct $i, j \in \{1, 2, 3\}$ such that $b_i <_\phi b_j <_\phi b$, then a_1, b_i, b_j, b would form an ordered K_4^- in ϕ. Thus, we conclude that there exist distinct $i, j \in \{1, 2, 3\}$ such that $b <_\phi b_i <_\phi b_j$, and arguing as before, we assume without loss of generality that the vertex b_1 occurs after every vertex of $V(G) \cup \{a_1, a_2, a_3, c_1, c_2, c_3\}$ in the ordering ϕ. Now if there exist distinct $i, j \in \{1, 2, 3\}$ such that $c <_\phi c_i <_\phi c_j$, then c, c_i, c_j, b_1 form an ordered K_4^- in ϕ. Thus, there exist distinct $i, j \in \{1, 2, 3\}$ such that $c_i <_\phi c_j <_\phi c$, and reasoning as before, we can assume without loss of generality that c_1 occurs before every vertex in $V(G) \cup \{b_1, b_2, b_3, d_1, d_2, d_3\}$. Using similar arguments, we conclude that d_1 occurs after every vertex in $V(G) \cup \{a_1, a_2, a_3, c_1, c_2, c_3\}$ in ϕ.

Claim *For every edge* $(u, v) \in E(G)$, *all the vertices in* $\{f_i^{(u,v)} : 1 \leq i \leq 6\}$ *occur between u and v in* ϕ.

Proof Suppose that there exists $i \in \{1, 2, \dots, 6\}$ such that $f_i^{(u,v)} <_\phi u <_\phi v$. Then the vertices $f_i^{(u,v)}, u, v, b_1$ form an ordered K_4^- in ϕ, which contradicts the fact that ϕ is a 3-C-E ordering. Similarly, if $u <_\phi v <_\phi f_i^{(u,v)}$ for some $i \in \{1, 2, \dots, 6\}$, then $a_1, u, v, f_i^{(u,v)}$ form an ordered K_4^- in ϕ; again a contradiction. □

Claim *For every edge* $e \in E(G)$ *and* $i \in \{1, 2, 3\}$, *the vertex* t_i^e *occurs between* f_{2i-1}^e *and* f_{2i}^e *in* ϕ.

Proof Since in the ordering ϕ, c_1 occurs before every vertex in $V(G)$ and d_1 occurs after every vertex in $V(G)$, it follows from the above claim that c_1 occurs before every vertex in F and d_1 occurs after every vertex in F. Now suppose that for some $e \in E(G)$ and $i \in \{1, 2, 3\}$, we have $f_{2i-1}^e, f_{2i}^e <_\phi t_i^e$. Then the vertices $c_1, f_{2i-1}^e, f_{2i}^e, t_i^e$ form an ordered K_4^- in ϕ, which is a contradiction. Similarly, if $t_i^e <_\phi f_{2i-1}^e, f_{2i}^e$, then the vertices $t_i^e, f_{2i-1}^e, f_{2i}^e, d_1$ form an ordered K_4^- in ϕ, again a contradiction. □

From the above two claims, it follows that for any edge $(a, b) \in E(G)$ and $i \in \{1, 2, 3\}$, the vertex $t_i^{(a,b)}$ occurs between a and b in ϕ. Now suppose for the sake of contradiction that $(u, v), (w, x), (y, z) \in E(G)$ form a disjoint triple in $\phi|_{V(G)}$, where $u <_\phi v \leq_\phi w <_\phi x \leq_\phi y <_\phi z$. Then we have $u <_\phi t_1^{(u,v)} <_\phi v \leq_\phi w <_\phi t_2^{(w,x)}, t_3^{(w,x)} <_\phi x \leq_\phi y <_\phi t_1^{(y,z)} <_\phi z$. But then the vertices $t_1^{(u,v)}, t_2^{(w,x)}, t_3^{(w,x)}, t_1^{(y,z)}$ form an ordered K_4^- in ϕ, a contradiction. □

Lemma 9 *If G is* 3*-colourable then* G' *has a* 3*-C-E ordering*.

Proof Let $M = V(G) \cup F$, $L = M \cup A \cup B \cup C \cup D$ and $U = F \cup T$. Note that $L \cap U = F$ and in fact F separates L and U. The idea is to apply Lemma 1 on the subgraphs L and U. Let V_1, V_2, V_3 be a partition of $V(G)$ into three colour classes. Let $E_1 = \{(u, v) \in E(G) \mid u \in V_1\}$ and $E_2 = \{(u, v) \in E(G) \mid u \in V_2, v \in V_3\}$. Observe that $\{E_1, E_2\}$ is a partition of $E(G)$. Let σ be an ordering of M such that

$$\sigma \in V_1 + \sum_{i \in 3,2,1} \sum_{e \in E_1} (f_{2i}^e, f_{2i-1}^e) + V_2 + \sum_{i \in 1,2,3} \sum_{e \in E_2} (f_{2i-1}^e, f_{2i}^e) + V_3$$

Here the sum notation denotes iterated concatenation. For example, the term $\sum_{i \in 3,2,1} \sum_{e \in E_1} (f_{2i}^e, f_{2i-1}^e)$ would be a shorthand for

$$(f_6^{e_1}, f_5^{e_1}, f_6^{e_2}, f_5^{e_2}, \dots, f_4^{e_1}, f_3^{e_1}, f_4^{e_2}, f_3^{e_2}, \dots, f_2^{e_1}, f_1^{e_1}, f_2^{e_2}, f_1^{e_2}, \dots)$$

where $\{e_1, e_2, \dots\}$ are the edges in E_1. Similarly, the term $\sum_{i \in 1,2,3} \sum_{e \in E_2} (f_{2i-1}^e, f_{2i}^e)$ would be

$$(f_1^{e^1}, f_2^{e^1}, f_1^{e^2}, f_2^{e^2}, \dots, f_3^{e^1}, f_4^{e^1}, f_3^{e^2}, f_4^{e^2}, \dots, f_5^{e^1}, f_6^{e^1}, f_5^{e^2}, f_6^{e^2}, \dots)$$

where $\{e^1, e^2, \dots\}$ are the edges in E_2. Let ϕ be the following ordering of L.

$$\phi = (a_1, a_2, a_3, a, c_1, c_2, c_3, c) + \sigma + (d, d_1, d_2, d_3, b, b_1, b_2, b_3)$$

We also define an ordering ψ of U as follows.

$$\psi = \sum_{i \in 3,2,1} \sum_{e \in E_1} (f^e_{2i}, t^e_i, f^e_{2i-1}) + \sum_{i \in 1,2,3} \sum_{e \in E_2} (f^e_{2i-1}, t^e_i, f^e_{2i})$$

Note that $\psi|_F = \sigma|_F = \phi|_F$. We claim that ϕ is a 3-C-E ordering of $G'[L]$ and ψ is a 3-C-E ordering of $G'[U]$. If the claim is true, we are done as the lemma statement will follow from an application of Lemma 1. Note that ψ has the property that, if $a \in T$ is adjacent to a 2-clique (u, v) in F, then $u <_\psi a <_\psi v$. Thus ψ satisfies the last condition for Lemma 1.

Claim *ψ is a 3-C-E ordering of $G'[U] = G'[T \cup F]$.*

Proof Suppose that there exists an ordered K_4^- having vertices u, x, y, v, such that $u <_\psi x <_\psi y <_\psi v$. Since every vertex in F has degree 2 in $G'[T \cup F]$, only the endpoints of $\{u, x, y, v\}$ can belong to F. Thus $\{x, y\} \subseteq T$. Now, the fact that every vertex in F is adjacent to exactly one vertex in T implies that $\{u, x, y, v\} \subseteq T$. Since $(u, v) \notin E(G')$, we have that $u, v \in \bigcup_{e \in E(G)} \{t^e_1\}$, which implies that $x, y \in \bigcup_{e \in E(G)} \{t^e_2, t^e_3\}$. But in ψ, it is impossible for two vertices from $\bigcup_{e \in E(G)} \{t^e_2, t^e_3\}$ to occur between two vertices in $\bigcup_{e \in E(G)} \{t^e_1\}$. Thus we have a contradiction. □

Claim *σ is a 3-C-E ordering of $G'[M] = G'[V(G) \cup F]$.*

Proof Suppose that there exists an ordered K_4^- having vertices u, x, y, v such that $u <_\sigma x <_\sigma y <_\sigma v$. Note that $\sigma|_{V(G)} \in V_1 + V_2 + V_3$, and it is therefore a 3-C-E ordering of $V(G)$ by Observation 3. Thus $\{u, x, y, v\}$ is not contained in $V(G)$, implying that $\{u, x, y, v\} \cap F \neq \emptyset$. Since $G'[F]$ is a collection of disjoint edges, it follows that $|\{u, x, y, v\} \cap F| \leq 2$. If F intersects $\{u, x, y, v\}$ at a single vertex, since every vertex in F has at most 2 neighbours in $V(G)$, that vertex must be an endpoint of $\{u, x, y, v\}$ and its neighbours in $V(G)$ must be x, y, a contradiction to the fact that every vertex in F lies between its two neighbours in $V(G)$ in σ. Thus $|\{u, x, y, v\} \cap F| = 2$. If the two vertices in $\{u, x, y, v\} \cap V(G)$ occur consecutively in u, x, y, v, then they have common neighbour in F that does not lie between them in σ, which is a contradiction. If in u, x, y, v, there is exactly one vertex in F between the two vertices in $V(G)$, then it means that in σ, there is a vertex in $V(G)$ between two adjacent vertices in F, which is again a contradiction. Thus we have that $x, y \in F$ and $u, v \in V(G)$. But now u and v are two neighbours of $x \in F$ in $V(G)$ that are not adjacent to each other, which is again a contradiction. □

Claim *ϕ is a 3-C-E ordering of $G'[L] = G'[M \cup A \cup B \cup C \cup D]$.*

Proof Suppose that there exists an ordered K_4^- having vertices u, x, y, v such that $u <_\phi x <_\phi y <_\phi v$. As a has no neighbours to its right and no non-neighbours to its left in ϕ, we have $a \notin \{u, x, y, v\}$. Symmetrically, $b \notin \{u, x, y, v\}$. If $c \in \{u, x, y, v\}$, then since in ϕ, c has no neighbours to its right and its only neighbours to the left are c_1, c_2, c_3, we have $v = c$, $u \in \{a_1, a_2, a_3\}$ and $x, y \in \{c_1, c_2, c_3\}$. But now we have a contradiction to the fact that $(u, x) \in E(G)$. Thus $c \notin \{u, x, y, v\}$. Symmetrically, we also get $d \notin \{u, x, y, v\}$. If $a_i, a_j \in \{u, x, y, v\}$ for some distinct $i, j \in \{1, 2, 3\}$, then $\{a_i, a_j\} = \{x, y\}$, since a_i and a_j are true twins. But then in ϕ, there is no common neighbour of a_i and a_j that is not a true twin of theirs to the left of x, so u has to be a true twin of x and y, which is a contradiction. Therefore, at most one among a_1, a_2, a_3 can be present in $\{u, x, y, v\}$, so we shall assume without loss of generality that $a_2, a_3 \notin \{u, x, y, v\}$. Using similar arguments, we shall assume without loss of generality that $b_2, b_3 \notin \{u, x, y, v\}$, $c_2, c_3 \notin \{u, x, y, v\}$ and $d_2, d_3 \notin \{u, x, y, v\}$.

First, suppose $c_1 \in \{u, x, y, v\}$, then since c_1 has no neighbours to the left of it in ϕ, we have $c_1 = u$, and $v \in \{b, d\}$ as they are the only non-neighbours of c_1 to the right of it. This is a contradiction to our earlier observation that $b, d \notin \{u, x, y, v\}$. Thus $c_1 \notin \{u, x, y, v\}$. Symmetrically, $d_1 \notin \{u, x, y, v\}$.

Now suppose that $a_1 \in \{u, x, y, v\}$. Then since a_1 has no neighbours to its left, we have $u = a_1$. As $(u, v) \notin E(G')$, we now have $v \in F$. Also, since x and y are neighbours of u, we have $x, y \in V(G) \cup \{b_1\}$. Since x and y occur before v in the ordering, we further have that $x, y \in V(G)$. But now we have the contradiction that in ϕ, the vertex $v \in F$ does not lie between its two neighbours $x, y \in V(G)$. Thus $a_1 \notin \{u, x, y, v\}$. Symmetrically, we have $b_1 \notin \{u, x, y, v\}$. Thus no vertex from $A \cup B \cup C \cup D$ can be in $\{u, x, y, v\}$, which implies that $\{u, x, y, v\} \subseteq M$. But this means that u, x, y, v form an ordered K_4^- in σ, which a contradiction to the previous claim. □

This proves that if G is 3-colourable, then G' has a 3-C-E ordering. □

Lemmas 8 and 9 prove the correctness of the reduction and thus we have the following theorem.

Theorem 6 *Find 3-C-E Ordering is NP-hard.*

7 $(n - k)$-C-E Ordering Parameterized by k

Observe that every graph on n vertices is an n-C-E graph. So following the lines of 'below-guarantee parameterization' [9], a natural problem to consider is deciding whether an input graph on n vertices has an $(n - k)$-C-E ordering when parameterized by k, and this has been called sometimes as *dual parameterization*. It is known for example that k-coloring is NP-hard for $k \geq 3$, but determining whether a graph on n vertices has a proper $(n - k)$-coloring is fixed-parameter tractable when parameterized by k (see [3]). We explore both the problems Verify $(n - k)$-C-E Ordering and Find $(n - k)$-C-E Ordering, parameterized by k, and show that they are fixed parameter tractable.

Theorem 7 *Verify $(n - k)$-C-E Ordering can be solved in time $2^k \cdot n^{O(1)}$.*

Proof Let G be the input graph and let ϕ be input ordering of $V(G)$. We can enumerate all maximal cliques on at least $n-k-1$ vertices in time $2^k \cdot n^{O(1)}$. To see this, observe that $C \subseteq V(G)$ is a clique on s vertices in G if and only if C is an independent set on s vertices in $\overline{G}$, which happens if and only if $V(G)\backslash C$ is a vertex cover in $\overline{G}$ on $n-s$ vertices. So it suffices to enumerate all minimal vertex covers in $\overline{G}$ on at most $k+1$ vertices. This can be done in $2^k \cdot n^{O(1)}$ time using the standard parameterized branching algorithm for vertex cover (see [3]). Once we have enumerated all maximal cliques in G on at least $n-k-1$ vertices, for each such clique C and pair of non-adjacent vertices $u, v \in V(G)$, we compute the set of vertices $S = \{s \in C \cap N(u) \cap N(v) \mid u <_\phi s <_\phi v\}$. If $|S| \geq n-k-1$, then $\{u, v\} \cup S$ contains an ordered K^-_{n-k+1} in ϕ and we can output "no". Otherwise, if $|S| < n-k-1$ for every clique C and pair of non-adjacent vertices u, v, then there is no ordered K^-_{n-k+1} in ϕ and we output "yes". □

Theorem 8 *Find* $(n-k)$*-C-E Ordering can be solved in time* $(3k^2)! \cdot 2^k \cdot n^{O(1)}$.

Proof Let G be the input graph. First, we prove the following claim.

Claim *If* $k^2 + 2k \leq n$ *then there is always an* $(n-k)$*-C-E ordering of* G, *and moreover, such an ordering can be found in time* $2^k \cdot n^{O(1)}$.

Proof Let C be an $(n-k)$-clique in G (if such a clique does not exist, then any ordering is an $(n-k)$-C-E ordering of G). Let $K = V(G)\backslash C$ and let $K' = \{v \in K : |N(v) \cap C| \geq n-2k\}$ be the set of vertices in K that have at least $n-2k$ neighbours in C. Note that $|K'| \leq |K| = k$. Let $S = \{u \in C : (u, v) \notin E(G) \text{ for some } v \in K'\}$ be the set of vertics in C that have a non-neighbour in K'. Since each $v \in K'$ has at least $n-2k$ neighbours in C, it follows that each $v \in K'$ has at most $n-k-(n-2k) = k$ non-neighbours in C, and so we have $|S| \leq |K'| \cdot k \leq k^2$.

Consider an ordering $\phi \in K + S + (C\backslash S)$. We claim that ϕ is an $(n-k)$-C-E ordering of G. Suppose for contradiction that ϕ is not an $(n-k)$-C-E ordering. Then there exists $Q \subseteq V(G)$ that induces an ordered K^-_{n-k+1} in ϕ. Let a, b be the endpoints of Q in ϕ so that $a <_\phi b$ and $(a, b) \notin E(G)$. Let $Q' = Q\backslash\{a, b\}$ so that Q' is an $(n-k-1)$-clique, a comes before Q' in ϕ, and b comes after Q' in ϕ. Since $|\{a\} \cup V(Q')| = n-k$, and since $|K| + |S| = k + k^2 \leq n-k$, it follows that b, which comes after $\{a\} \cup V(Q')$ in ϕ, must come after both K and S in ϕ, and thus lies in $C\backslash S$. Since $(a, b) \notin E(G)$, and since C is a clique, it follows that $a \in V(G)\backslash C = K$. Moreover, since a is adjacent to every vertex in Q', it follows that a has at least $n-k-1$ neighbours. But since $a \in K$, at most $|K| - 1 = k-1$ of these neighbours can be from K. It follows that at least $n-2k$ neighbours of a must lie in C. Therefore we have $|N(a) \cap C| \geq n-2k$ and thus $a \in K'$. Now, by definition of S, and the fact that $b \notin S$, it follows that $(a, b) \in E(G)$, a contradiction. Thus we conclude that ϕ contains no ordered K^-_{n-k+1}, or in other words, ϕ is an $(n-k)$-C-E ordering of G.

As explained in the Proof of Theorem 7, an $(n-k)$-clique C in G can be found in $2^k \cdot n^{O(1)}$ time[1] and therefore it is clear that the ordering ϕ can also be constructed in $2^k \cdot n^{O(1)}$ time. □

Now we give an algorithm for Find $(n-k)$-C-E Ordering that runs in time $(3k^2)! \cdot 2^k \cdot n^{O(1)}$. If $k^2 + 2k \leq n$ then we use the above claim to find an $(n-k)$-C-E ordering of G in time $2^k \cdot n^{O(1)}$. Otherwise $n < k^2 + 2k$, which implies $3k^2 > n$. Hence we can enumerate all $n! < (3k^2)!$ orderings of $V(G)$ and check if each of them are $(n-k)$-C-E orderings, which can be done in time $2^k \cdot n^{O(1)}$ using the verification algorithm in Theorem 7. Thus the total running time is $(3k^2)! \cdot 2^k \cdot n^{O(1)}$. □

8 Conclusions and Open Problems

We have shown that the problem of determining whether a given graph has a k-C-E ordering is NP-hard for each $k \geq 3$ settling an open problem in the literature. A natural open problem is to identify graph classes where the problem can be solved in polynomial time.

Finding a maximum clique in a k-C-E graph on n vertices is known to have an $n^{O(k)}$ algorithm when a k-clique-extendible ordering is given, which we prove to be optimal under plausible conjectures. It is also an open problem mentioned before [14] whether we can find a maximum clique in a k-C-E graph in polynomial time for a fixed k, if given only the adjacency matrix of the graph. It would also be interesting to know polynomial time solvable problems in k-C-E graphs, even for $k = 3$. As triangle free graphs and diamond-free graphs are 3-C-E graphs, we know that the maximum independent set problem and the chromatic number problem are NP-hard in these classes of graphs.

Observing that every graph on n vertices has a n-C-E ordering, we have shown that determining whether the graph has a $(n-k)$-C-E ordering is fixed-parameter tractable when parameterized by k. Actually, every graph has a $(\omega(G)+1)$-C-E ordering where $\omega(G)$ is the number of vertices in a maximum clique in G. So an interesting open problem is the parameterized complexity of finding a $(\omega(G)+1-k)$ -C-E ordering parameterized by k.

It would also be interesting to study whether k-C-E graphs can be recognised approximately. There are two suitable notions for approximation. One is the following: An algorithm is said to be an α-factor approximation (for $\alpha \geq 1$) if, given a graph G and integer k, it either outputs a (αk)-C-E ordering or concludes that no k-C-E ordering exists for G. The second notion is the following: An algorithm is said to be a α-factor approximation (for $\alpha \leq 1$) if, given a graph G and integer k, outputs an ordering ϕ such that at most α fraction of the induced K_{k+1}^- in the graph are

[1] Unlike in Theorem 7, here we only need to find just a clique on $(n-k)$ vertices, for which faster algorithms exist, but as the runtime for the other case dominates the overall runtime, a $2^k n^{O(1)}$ algorithm suffices.

ordered in ϕ. Note that solving this problem for $\alpha = 0$ is equivalent to solving Find k -C-E Ordering.

For the second notion of approximation, there is an easy $\left(\frac{2}{k(k+1)}\right)$-factor approximation. Simply output a random ordering of the vertices of G. The probability that any given induced K_{k+1}^- is ordered is $\frac{2}{k(k+1)}$. Thus by linearity of expectation, a $\frac{2}{k(k+1)}$ fraction of all the induced K_{k+1}^- in G will be ordered.

Finally as k-C-E graphs are defined by orderings, it would be interesting to relate k-C-E graphs to other graphs defined by (vertex or edge) orderings [2, Chapter 5] and explore interesting properties about them along the lines of those graphs defined by orderings.

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