# A MODEL-THEORETIC PERSPECTIVE ON ALGEBRAIC REDUCTIONS OF HYPERKAEHLER MANIFOLDS

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Abstract. A model-theoretic abstraction and treatment of the material in Section 2 of [Campana, Oguiso, and Peternell. Non-algebraic hyperkähler manifolds. *Journal of Differential Geometry*, 85(3):397–424, 2010] is given. The property of a finite rank type having all nontrivial fibrations almost internal to a given invariant set of types is introduced and explicated. As a consequence, on the complex-analytic side, it is shown that the algebraic reduction of a nonalgebraic (generalised) hyperkähler manifold does not descend.

## 1. Introduction

In section 2 of [1], Campana, Oguiso, and Peternell give a certain structure theorem for hyperkähler manifolds, in fact for compact complex manifolds of dimension $2n$ with $h^{2,0} = 1$ and the corresponding holomorphic 2-form satisfying $\sigma^n \neq 0$. These are sometimes called *generalised hyperkähler manifolds*. The starting point is the following essential property of such a manifold $X$ (this is Theorem 2.3(1) of [1]):

(1) If $f : X \to Y$ is a fibration with $\dim(Y) < \dim(X)$ then $Y$ is Moishezon.

An immediate consequence of (1) is:

(2) The algebraic reduction $f : X \to B$ is minimal in the sense that it has no proper intermediate fibrations.

And from (2) they deduce:

(3) The generic fibre of $f : X \to B$ is either Moishezon, in which case it is an abelian variety, or of algebraic dimension zero in which case it is "isotypically semi-simple".

Our aims are firstly to give a tighter (including model-theoretic/stability-theoretic) account of the property (1), and secondly to deduce (3) from (2) on general model-theoretic grounds.

As far as the first aim is concerned, we show (see Proposition 4.6) that (1) is equivalent to the algebraic reduction $X \to B$ map being minimal AND not almost descending to any $S$ with $\dim(S) < \dim(B)$ (assuming $X$ non Moishezon). This notion of (almost) descent is the natural one in bimeromorphic geometry, and specialises a general notion of "descent for types" in model theory (see Definition 3.4). In fact, the characterisation of (1) just described is obtained in a general model-theoretic setting (see Proposition 3.6).

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Concerning the second aim, we succeed (see Proposition 4.2) in recovering model-theoretically from (2) the dichotomy between Moishezon versus isotypically semi-simple algebraic dimension zero fibres. See Proposition 2.3 for the general model-theoretic result. To prove the rest of (3), namely that in the Moishezon case the fibre is an abelian variety, we use definable automorphism groups as well as a result of Fujiki. This is done in Proposition 4.3.

The contributions of this paper, then, are some new notions and results on the stability-theoretic side motivated by complex geometry, as well as, on the complex-analytic side, the role of descent in the explication of property (1).

In sections 2 and 3, we work in the general setting of a sufficiently saturated model $\overline{M}^{\mathrm{eq}}$ of a complete theory $T$ of finite $U$-rank. In section 4, we obtain our conclusions for compact complex manifolds (with some additional arguments). Our stability-theoretic notation is standard, see [9]. For elements of complex geometry and the translation between this and model theory, see for example [7] and [8].

## 2. No Fibrations

In the next two sections we work in the general setting of a sufficiently saturated model $\overline{M}^{\mathrm{eq}}$ of a complete theory $T$ of finite $U$-rank.

**Definition 2.1.** A stationary type $\mathrm{tp}(a/A)$ *admits no proper fibrations* if whenever $c \in \mathrm{dcl}(Aa) \setminus \mathrm{acl}(A)$ then $a \in \mathrm{acl}(Ac)$.

Clearly, any minimal type admits no proper fibrations. But there are others: in [1] it is pointed out that if $S$ is a simple K3 surface, then the generic type of the symmetric power $S^{[2]}$, which is of rank 2 as it is the finite-to-one image of $S \times S$, admits no proper fibrations. In fact, generalising this example we have:

**Lemma 2.2.** *Suppose $p \in S(A)$ is a trivial minimal stationary type. If $a_1, \dots, a_r$ are independent realisations of $p$ and $c$ is a code for $\{a_1, \dots, a_r\}$, then $\mathrm{tp}(c/A)$ admits no proper fibrations.*

*Proof.* Let $d \in \mathrm{dcl}(Ac)$. We wish to show that either $d \in \mathrm{acl}(A)$ or $c \in \mathrm{acl}(Ad)$. Suppose some $a_i \in \mathrm{acl}(Ad)$. Then, as every permutation of $\{a_1, \dots, a_r\}$ extends to an automorphism fixing $Ac$, and hence fixing $Ad$, it follows that each $a_1, \dots, a_r \in \mathrm{acl}(Ad)$. But then $c \in \mathrm{acl}(Ad)$, and we are done. We may therefore assume that each $a_i \notin \mathrm{acl}(Ad)$. By triviality, it follows that $d \mathop{\smile\!\!\!\!\!\mid}_A (a_1, \dots, a_r)$. But $d \in \mathrm{acl}(Aa_1 \dots a_r)$, and hence $d \in \mathrm{acl}(A)$, as desired. □

Recall that the following are equivalent for any stationary type $p = \mathrm{tp}(a/A)$:

(i) $p$ is semiminimal

(ii) $p$ is almost internal to some minimal type.

(iii) There exists $B \supseteq A$ a minimal type $r$ over $B$ and a $B$-independent tuple $(e_1, \dots, e_\ell)$ of realisations of $r$, such that $a \mathop{\smile\!\!\!\!\!\mid}_A B$ and $a$ is interalgebraic with $(e_1, \dots, e_\ell)$ over $B$.

**Proposition 2.3.** *If $p = \mathrm{tp}(a/A)$ is stationary and admits no proper fibrations then $p$ is semiminimal. In fact, under this hypothesis, either $p$ is $r$-semiminimal for some nonmodular minimal $r$, or $a$ is interalgebraic over $A$ with a tuple of realisations of some modular minimal type over $A$.*

*Proof.* We may assume that $p$ is nonalgebraic. Hence $p$ is nonorthogonal to some minimal type $r$. Let $\mathbf{R}$ be the set of $\mathrm{acl}(A)$-conjugates of $r$. Then, as $p$ is not foreign to $\mathbf{R}$, there exists $c \in \mathrm{dcl}(Aa) \setminus \mathrm{acl}(A)$ with $\mathrm{stp}(c/A)$ $\mathbf{R}$-internal (see 7.4.6 of [9]). By the assumption of having no proper fibrations we must have $a \in \mathrm{acl}(Ac)$, so that $\mathrm{stp}(a/A) = p$ is almost $\mathbf{R}$-internal. But this implies that $p$ is almost internal to $r$, and so $r$-semiminimal.

Suppose now that $r$ is modular. Then, with $c$ as above, we have that $\mathrm{stp}(c/A)$ is 1-based. If $U(c/A) = 1$ then, as $a$ is interalgebraic with $c$ over $A$, we have that $p$ itself is minimal modular, and we are done. If $U(c/A) > 1$ then let $e$ be such that $U(c/Ae) = U(c/A) - 1$, and $e = \mathrm{cb}(c/Ae)$. By 1-basedness, $e \in \mathrm{acl}(Ac)$, and hence the Lascar inequalities yield that

$$U(e/A) = U(c/A) + U(e/Ac) - U(c/Ae) = 1$$

So $\mathrm{stp}(e/A)$ is modular and minimal. Now let $(e_1, \dots, e_n)$ be the $Ac$-conjugates of $e$, so that $(e_1, \dots, e_n) \in \mathrm{dcl}(Ac) \subseteq \mathrm{dcl}(Aa)$. The assumption of no fibrations yields that $a \in \mathrm{acl}(Ae_1, \dots, e_n)$. So $(e_1, \dots, e_n)$ is a finite tuple of realisations of a modular minimal type over $A$, and $a$ is interalgebraic with this tuple over $A$. □

*Remark* 2.4. As we will see in §4 below, the specialisation of the above proposition to bimeromorphic geometry yields a uniform version of Theorem 2.4(1) of [1]. However, the proof in [1] is not a specialisation of ours. In the non-Moishezon case, for example, it does not go via a minimal modular type, but rather only uses the orthogonality of $p$ to the projective line. The authors establish first a certain fact about maximal covering families of algebraic dimension zero compact Kähler manifolds (Lemma 2.7 of [1]) which is if independent interest, and which we can explain model-theoretically as the follows: *Suppose $T$ is the theory of a Zariski structure*[1] *with the* CBP[2]*, and $p = \mathrm{stp}(a/A)$ is orthogonal to the set of nonmodular minimal types. Let $q = \mathrm{stp}(a/E)$ be a forking extension of $p$ with maximal locus.*[3] *Let $e = \mathrm{cb}(q)$. Then $e \in \mathrm{acl}(Aa)$ and $U(e/A) = 1$.*

*Proof.* First we note, without using the CBP, that $q$ being a forking extension of $p$ of maximal locus is equivalent to $U(q) = U(p) - 1$. The right to left direction is clear. For the converse, note that $e \notin \mathrm{acl}(A)$ and hence there is a $B \supset A$ such that $U(e/B) = 1$. Let $q' = \mathrm{stp}(a'/Be)$ be the nonforking extension of $q$ to $Be$. Since $e = \mathrm{cb}(q')$ and $e \notin \mathrm{acl}(B)$, $a' \not\perp_B e$. That is, $e \in \mathrm{acl}(Ba')$. By Lascar inequalities it follows that $U(a'/B) = U(a'/Be) + 1 = U(q) + 1$. So if $U(q) < U(p) - 1$ then $\mathrm{stp}(a'/B)$ is a forking extension of $p$ which has $q'$ as a forking extension. The locus of $\mathrm{stp}(a'/B)$ thus properly contains $\mathrm{loc}(q') = \mathrm{loc}(q)$, which is a contradiction to maximality.

So $U(q) = U(p) - 1$. Now if $e \notin \mathrm{acl}(Aa)$, then letting $e_0 = \mathrm{acl}(Ae) \cap \mathrm{acl}(Aa)$ we would have $a \not\perp_{e_0} e$ since $e = \mathrm{cb}(a/e)$. So $U(a/e_0) > U(a/e) = U(p) - 1$ which forces $a \perp_A e_0$, and so $\mathrm{stp}(a/e_0)$ is still orthogonal to the set of nonmodular minimal types. On the other hand, CBP tells us that $\mathrm{stp}(e/e_0)$ is almost internal to the set of nonmodular minimal types (this is Theorem 2.1 of [2] but see also Proposition 4.4 of [8]). This contradicts $a \not\perp_{e_0} e$. Hence $e \in \mathrm{acl}(Aa)$. That $U(e/A) = 1$ now follows immediately from the Lascar inequalities. □

[1]This is in the sense of Zilber [11].

[2]This is the"canonical base property", see [8] for details.

[3]The *locus* of $\mathrm{tp}(a/E)$ is the smallest Zariski-closed set defined over $E$ containing $a$.

## 3. All Fibrations over $\mathbb{P}$

Fix an $A$-invariant set of partial types, $\mathbb{P}$. Often we are interested in the case when $\mathbb{P}$ is the set of all nonmodular minimal types; in $\mathrm{DCF}_0$ we usually take $\mathbb{P}$ to be the field of constants and in CCM it is the sort of the projective line. But the material in this section makes sense for any $\mathbb{P}$.

**Definition 3.1.** Suppose $p := \mathrm{tp}(a/A)$ is stationary. We say that *all fibrations of $p$ are over* $\mathbb{P}$ if whenever $c \in \mathrm{dcl}(Aa)$ with $a \notin \mathrm{acl}(Ac)$, then $\mathrm{stp}(c/A)$ is almost $\mathbb{P}$-internal.

The generic type of a generalised hyperkähler manifold in CCM has this property (with $\mathbb{P}$ the complex projective line); this is Theorem 2.3(1) of [1]. In $\mathrm{DCF}_0$, with $\mathbb{P}$ the field of constants, an example is given by the generic type of the $\delta$-variety given by $\delta\big(\frac{\delta x}{x}\big) = 0$; indeed, as a consequence of the Zilber trichotomy in $\mathrm{DCF}_0$, any rank 2 type that is analysable in the constants will have all fibrations over the constants.

**Definition 3.2** ($\mathbb{P}$-reduction)**.** Let $\mathrm{Int}_A(\mathbb{P}) := \{c : \mathrm{stp}(c/A) \text{ is almost } \mathbb{P}\text{-internal}\}$. We say that $b$ is the *$\mathbb{P}$-reduction of $a$ over $A$* if $\mathrm{dcl}(Ab) = \mathrm{dcl}(Aa) \cap \mathrm{Int}_A(\mathbb{P})$.

*Remark* 3.3.
1. All fibrations of $\mathrm{tp}(a/A)$ being over $\mathbb{P}$ can be reformulated as an "exchange" property: if $c \in \mathrm{dcl}(Aa) \setminus \mathrm{Int}_A(\mathbb{P})$ then $a \in \mathrm{acl}(Ac)$.
2. The $\mathbb{P}$-reduction of $a$ over $A$ is precisely the canonical base of $\mathrm{tp}\big(a/\,\mathrm{Int}_A(\mathbb{P})\big)$ and has the property that $\mathrm{tp}\big(a/\,\mathrm{dcl}(Aa) \cap \mathrm{Int}_A(\mathbb{P})\big) \vdash \mathrm{tp}\big(a/\,\mathrm{Int}_A(\mathbb{P})\big)$. See, for example, Lemma 1 of the appendix to [3].

We will characterise when $\mathrm{tp}(a/A)$ has all fibrations over $\mathbb{P}$ in terms of the structure of the extension $\mathrm{tp}(a/Ab)$ where $b$ is the $\mathbb{P}$-reduction of $a$ over $A$. The key structural property in this characterisation will be the following natural notion of descent for types, which is the model-theoretic counterpart to birational descent in algebraic geometry and bimeromorphic descent in complex-analytic geometry (see Definition 4.4 below).

**Definition 3.4** (Descent for types)**.** A complete stationary type $p \in S(B)$ *descends* to $B_0 \subseteq \mathrm{dcl}(B)$ if there exist a stationary $q \in S(B_0)$ and a $B$-definable bijection $f : p^{\overline{M}} \to q_B^{\overline{M}}$, where by $q_B$ we mean the nonforking extension of $q$ to $B$. If the map $f$ is only assumed to be finite-to-one and surjective, then we say that $p$ *almost descends* to $B_0$.

*Remark* 3.5.
1. A special case of $p \in S(B)$ descending to $B_0 \subseteq \mathrm{dcl}(B)$, is when $p$ does not fork over $B_0$ and $p \upharpoonright B_0$ is stationary.
2. Suppose $p = \mathrm{tp}(a/Ab)$ where $b$ is the $\mathbb{P}$-reduction of $a$ over $A$. If $p$ descends to $Ab_0$ witnessed by $q \in S(Ab_0)$, then in fact $q$ is weakly orthogonal to $\mathrm{Int}_A(\mathbb{P})$. In particular, $q$ has a unique extension to $Ab$ and we have an $Ab$-definable bijection between $p^{\overline{M}}$ and $q^{\overline{M}}$.

*Proof of 3.5(2).* Let $q = \mathrm{tp}(a_0/Ab_0)$ with $a_0 \mathop{\smile\!\!\!\!\!\!\mid}_{Ab_0} b$. Since $a \mathop{\smile\!\!\!\!\!\!\mid}_{Ab} \mathrm{Int}_A(\mathbb{P})$ and $a_0 \in \mathrm{dcl}(Aba)$, we have $a_0 \mathop{\smile\!\!\!\!\!\!\mid}_{Ab} \mathrm{Int}_A(\mathbb{P})$. By transitivity we get that $a_0 \mathop{\smile\!\!\!\!\!\!\mid}_{Ab_0} \mathrm{Int}_A(\mathbb{P})$. This implies, by invariance, that $q$ is weakly orthogonal to $\mathrm{Int}_A(\mathbb{P})$. Using the fact that $b$ is from $\mathrm{Int}_A(\mathbb{P})$, we get the "in particular" clause. $\square$

Here is our characterisation of having all fibrations over $\mathbb{P}$.

**Proposition 3.6.** *Suppose $p = \mathrm{tp}(a/A)$ is stationary and not almost internal to $\mathbb{P}$, and $b$ is the $\mathbb{P}$-reduction of $a$ over $A$. Then the following are equivalent:*

(i) *all fibrations of $p$ are over $\mathbb{P}$,*
(ii) *$\mathrm{tp}(a/Ab)$ admits no proper fibrations and does not almost descend to any relatively algebraically closed proper subset of $\mathrm{dcl}(Ab)$ containing $A$.*

*Proof.* We begin with the following characterisation for descent of the $\mathbb{P}$-reduction.

**Claim 3.7.** *$\mathrm{tp}(a/Ab)$ almost descends to $Ab_0$ if and only if there exists $a_0 \in \mathrm{dcl}(Aa)$ such that $a \in \mathrm{acl}(Aba_0)$ and $\mathrm{dcl}(Ab_0)$ contains the $\mathbb{P}$-reduction of $a_0$ over $A$.*

*Proof of Claim 3.7.* If $a_0 \in \mathrm{dcl}(Aa)$ and $a \in \mathrm{acl}(Aba_0)$ then there exists an $Ab$-definable finite-to-one surjection $\mathrm{tp}(a/Ab)^{\overline{M}} \to \mathrm{tp}(a_0/Ab)^{\overline{M}}$. If, moreover, $\mathrm{dcl}(Ab_0)$ contains the $\mathbb{P}$-reduction of $a_0$ over $A$, then, as $b$ is from $\mathrm{Int}_A(\mathbb{P})$, we have $a_0 \mathop{\smile\!\!\!\!\!\!\;|}_{Ab_0} b$ and $\mathrm{tp}(a_0/Ab_0)$ is stationary. Hence $\mathrm{tp}(a/Ab)$ almost descends to $Ab_0$, as desired.

Conversely, suppose $q \in S(Ab_0)$ and $f : \mathrm{tp}(a/Ab)^{\overline{M}} \to q_{Ab}^{\overline{M}}$ witnesses that $\mathrm{tp}(a/Ab)$ almost descends to $Ab_0$. Letting $a_0 := f(a)$ it suffices to show that $b_0$ contains the $\mathbb{P}$-reduction of $a_0$ over $A$. We know that $a_0 \mathop{\smile\!\!\!\!\!\!\;|}_{Ab_0} b$ since $q_{Ab}$ is the nonforking extension of $q$. On the other hand, as $b$ is the $\mathbb{P}$-reduction of $a$ over $A$, $a_0 \mathop{\smile\!\!\!\!\!\!\;|}_{Ab} \mathrm{Int}_A(\mathbb{P})$. Hence $a_0 \mathop{\smile\!\!\!\!\!\!\;|}_{Ab_0} \mathrm{Int}_A(\mathbb{P})$, and so $\mathrm{dcl}(Ab_0)$ contains the $\mathbb{P}$-reduction of $a_0$ over $A$. □

Next, we prove that (i) implies (ii) in Proposition 3.6. It follows more or less immediately from the definitions that since all fibrations of $p = \mathrm{tp}(a/A)$ are over $\mathbb{P}$, and $b$ is the $\mathbb{P}$-reduction of $a$ over $A$, that $\mathrm{tp}(a/Ab)$ admits no proper fibrations. So assuming that $\mathrm{tp}(a/Ab)$ descends to $Ab_0$ it remains for us to show that $b \in \mathrm{acl}(Ab_0)$. Let $a_0$ be as given by Claim 3.7. Since $p$ is assumed to not be almost internal to $\mathbb{P}$ and $a \in \mathrm{acl}(Aba_0)$, we must have $a_0 \notin \mathrm{Int}_A(\mathbb{P})$. By (i), $a \in \mathrm{acl}(Aa_0)$ and hence $b \in \mathrm{acl}(Aa_0)$. But $a_0 \mathop{\smile\!\!\!\!\!\!\;|}_{Ab_0} \mathrm{Int}_A(\mathbb{P})$ by the claim, so that $b \in \mathrm{acl}(Ab_0)$, as desired.

Conversely, suppose that (ii) holds and $c \in \mathrm{dcl}(Aa) \setminus \mathrm{Int}_A(\mathbb{P})$. We need to show that $a \in \mathrm{acl}(Ac)$. As $\mathrm{tp}(a/Ab)$ has no proper fibrations and $c \notin \mathrm{acl}(Ab)$, we must have $a \in \mathrm{acl}(Abc)$. Let $b_0$ be the $\mathbb{P}$-reduction of $c$ over $A$. By Claim 3.7 we have that $\mathrm{tp}(a/Ab)$ almost descends to $Ab_0$, and so by (ii), $b \in \mathrm{acl}(Ab_0) \subset \mathrm{acl}(Ac)$. So $a \in \mathrm{acl}(Ac)$, as desired. □

## 4. Consequences for CCM

In this section we discuss the above notions in the particular case of the theory CCM, from which, in any case, the ideas in this paper stem. As we shall see, we recover the results of §2 of [1] and even add a little to them.

First some preliminaries. By a *fibration* we will mean a dominant meromorphic map $f : X \to S$ between irreducible compact complex spaces whose general fibres are irreducible. This differs slightly from standard terminology in that we insist on irreducibility rather than connectedness (see [1], for example). Up to taking a normalisation of $X$ these notions will agree; and the reason for our stricter definition is the following characterisation.

**Fact 4.1** (Lemmas 2.7 and 2.11 of [6])**.** Work in a saturated model of CCM. Suppose $p = \mathrm{tp}(a/b)$, $X = \mathrm{loc}(a, b)$, $Y = \mathrm{loc}(b)$, and $\pi : X \to Y$ is the co-ordinate projection. Then the following are equivalent:

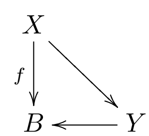
(i) The general fibres of $\pi$ are irreducible.
(ii) $p$ is stationary.
(iii) The generic fibre $X_b$ is absolutely irreducible.

Fibrations are well-behaved with respect to base change. If $f : X \to S$ is a fibration and $g : T \to S$ is another dominant meromorphic map between irreducible compact complex spaces then the Zariski closure of

$$\{(a, b) : \ f \text{ is defined at } a, \ g \text{ is defined at } b, \text{ and } f(a) = g(b)\}$$

in $X \times T$ has a unique irreducible component that projects onto $T$. The projection of this component onto $T$ is a fibration whose general fibres agree with that of $f$. We will denote this fibration by $X_{(T)} \to T$, and refer to it as the *strict pull back* of $X \to S$ in $X \times_S T \to T$. Note that $X_{(T)}$ also projects onto $X$ and is maximal dimensional in $X \times_S T$.

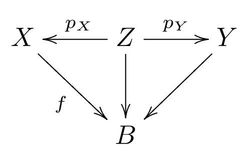
In [1] a fibration $f : X \to B$ is called *minimal* if whenever there is a factorisation




then either $\dim Y = \dim B$ or $\dim Y = \dim X$. If $X = \text{loc}(a)$ and $B = \text{loc}(b)$, then it is not hard to see that $f$ is minimal if and only if $p = \text{tp}(a/b)$ admits no proper fibrations in the sense of §2. Our Proposition 2.3 then implies the following slightly more uniform version of Theorem 2.4(1) of [1].

**Proposition 4.2.** *Suppose $X$ is an irreducible compact complex space of Kähler-type and $f : X \to B$ is minimal. Then either*

(I) *the general fibres of $f$ are Moishezon; or,*
(II) *the general fibres of $f$ are* uniformly isoptypically semi-simple *of algebraic dimension zero; that is, there is a commutative diagram*




*where $Y$ is of the form $Y' \times_B Y' \times_B \cdots \times_B Y'$ for some fibration $Y' \to B$ with general fibres simple of algebraic dimension zero, and $p_X, p_Y$ are generically finite surjective holomorphic maps.*

*Proof.* This is essentially a translation of 2.3 which tells us that one of two things can happen: One possibility is that the generic type of the generic fibre of $f$ is $r$-semiminimal for some nonmodular minimal type $r$. As a consequence of the manifestation of the Zilber dichotomy in CCM, this will imply that the general fibres of $f$ are Moishezon (see Fact 3.1 of [8] or Proposition 4.4 of [6]). The other possibility is that the generic type of the generic fibre of $f$ is interalgebraic with a finite tuple of (independent) realisations of a minimal modular type. We deduce from this the existence of the diagram in (II) where $Y' \to B$ has the property that the generic type of the generic fibre is minimal modular. As we are working in the Kähler case we have “essentially saturation” (see [7]), and so the *general* fibres of $Y' \to B$ will have minimal modular generic types. Minimality yields simplicity of the general fibres, and modularity forces them to have algebraic dimension zero (see Remark 3.3 of [10]). □

In the second part of Theorem 2.4 of [1] the authors use a structure theorem of Fujiki's (Theorem 1 of [4]) on the algebraic reduction of compound Moishezon spaces of Kähler-type to conclude further that, in the case when $f$ is the algebraic reduction, case (I) entails that the general fibre of $X \to B$ is an abelian variety.[4] We can make model-theoretic sense of this too.

**Proposition 4.3.** *Let $X$ be a non-Moishezon irreducible compact complex space of Kähler-type, and $f : X \to B$ the algebraic reduction. Suppose that $f$ is minimal. If the general fibre of $f$ is Moishezon then it is in fact an abelian variety.*

*Proof.* We use the theory of internality and follow roughly the ideas of the second author in [10] (Fact 5.1). Since we are in a Kähler-type space, we can work in a fixed full countable language for $X$ with respect to which our structure is saturated (see [7]). Let $a$ be generic in $X$, $b := f(a)$, and $V$ the set of realisations of $\mathrm{tp}(a/b)$. That the general fibre of $f$ is Moishezon implies that $V$ is internal to $\mathbb{P}(\mathbb{C})$, and hence by the theory of internality (cf. §7.4 of [9]) is $b$-definably and faithfully acted upon by a $b$-definable group $G$ that is definably isomorphic to a connected algebraic group. The fact that $f$ is the algebraic reduction implies that $a \perp_b \mathbb{P}(\mathbb{C})$, and hence the action is transitive. Now $G$ has a unique maximal definable subgroup that is definably isomorphic to a connected linear algebraic group, let's call it $L$. Note that $L$ must be $b$-definable and normal, and that the quotient $G/L$ is a $b$-definable group definably isomorphic to an abelian variety.

We first rule out the possibility that $L$ acts transitively on $V$. If it did, then the generic fibre $X_b$ of $f$ would be almost homogeneous unirational. This implies that $f$ is Moishezon (see Proposition 7 of [5], this is where Kähler-type is used). Since $B$ is Moishezon this would contradict $X$ being non-Moishezon. So $L$ does not act transitively on $V$.

Next we argue that $L$ must fix $a$; we show that $L$ moving $a$ is ruled out by the minimality of $f$. Let $c$ be a code for $L \cdot a$. So $c \in \mathrm{dcl}(a)$. Note that for any $a' \in V$, since the type of $a'$ and $a$ over $b$ agree, the code for $L \cdot a'$ has the same type as $c$ over $b$. Since $L$ does not act transitively it will have infinitely many orbits in $V$ (by stationarity), and so $c \notin \mathrm{acl}(b)$. On the other hand, note that if $a' \in L \cdot a$ then $a'$ and $a$ have the same type over $bc$. If $L$ does not stabilise $a$ the orbit $L \cdot a$ is infinite (by connectedness), and hence $a \notin \mathrm{acl}(bc)$. But $\mathrm{tp}(a/b)$ has no proper fibrations as $f$ is minimal, and that rules out the possibility of such a $c$.

It must therefore be the case that $L$ fixes $a$. Since $V$ is a complete type over $b$ and everything is $b$-definable, $L$ must stabilise all of $V$. By the faithfulness of the action, $L$ is thus trivial, and $G$ is definably isomorphic to some abelian variety $A \subseteq \mathbb{P}^n(\mathbb{C})$. We thus have a definable transitive action of $A$ on $V$.

It remains to show that this action of $A$ on $V$ is holomorphic. Indeed, note first of all that $V$ is a Zariski open subset of $X_b$; this follows from saturation because $V$ is a definable set (it is the orbit of $a$ under the action of $A$) and also an intersection of Zariski open subsets of $X_b$ (as $\mathrm{tp}(a/b)$ is generic in $X_b$). Hence the question

[4] In the statement of Theorem 2.4(2) of [1] the authors fail to mention the assumption that $f$ is the algebraic reduction. However, it is not hard to find counterexamples if we drop this assumption, and inspecting the proof shows that the authors have in mind the case when $X$ is hyperkähler nonalgebraic, in which case $f$ is the algebraic reduction.

of whether the action is holomorphic or not makes sense. If it were holomorphic then by quantifier elimination there is a meromorphic map $\phi : A \times X_b \to X_b$ that agrees with the action of $A$ on $A \times V$. But as the action is transitive, it follows that $V = \phi(A \times \{a\})$ is Zariski closed in $X_b$ and hence equal to $X_b$. So we have a transitive holomorphic action of $A$ on $X_b$, which forces $X_b$ to be an abelian variety.

So we prove that the action of $A$ on $V$ is holomorphic. First of all, because of the commutativity of $A$, the parameters over which $A$ and its action on $V$ are defined can be taken to be a tuple $b' \supseteq b$ from $\mathbb{P}(\mathbb{C})$. Now, by quantifier elimination, the action restricted to some nonempty $b'$-definable Zariski open subset $U \subseteq A \times V$ is holomorphic. But as $\operatorname{tp}(a/b) \vdash \operatorname{tp}(a/\mathbb{P})$ and both $b'$ and $A$ live in $\mathbb{P}(\mathbb{C})$, this $U$ can be taken to be of the form $A' \times V$ where $A'$ is a nonempty $b'$-definable Zariski open subset of $A$. It follows that for any $x \in A'$, the action of $A$ restricted to $(x + A') \times V$ is holomorphic as it is given by $(x + y, v) \mapsto x \cdot (y \cdot v)$. But such translates of $A'$ by elements of $A'$ cover all of $A$. So our action of $A$ on $V$ is holomorphic, as desired. $\square$

Next we consider the complex-analytic content of Proposition 3.6. The role of $\mathbb{P}$ here is played by the projective line sort. The first thing to notice is that $\mathbb{P}$-reductions in the sense of Definition 3.2 when specialised to CCM agree with algebraic reductions. That is, *$b$ is the $\mathbb{P}$-reduction of $a$ if and only if $b = f(a)$ where $f$ is the algebraic reduction map on $X := \operatorname{loc}(a)$.* Indeed, this follows from the fact that $\operatorname{tp}(b)$ is almost internal to $\mathbb{P}$ if and only if $\operatorname{loc}(b)$ is Moishezon – see Fact 3.1 of [8], for example.

We now recall bimeromorphic descent for fibrations.

**Definition 4.4** (Descent for fibrations)**.** Suppose $h : X \to T$ is a fibration and $g : T \to S$ is a dominant meromorphic map. We say that $h$ *descends to* $S$ if there exists a fibration $\hat{h} : \widehat{X} \to S$, such that $X$ is bimeromorphically equivalent to $\widehat{X}_{(T)}$ over $T$. In diagrams

$$\begin{array}{ccccc} X & \xrightarrow{\approx} & \widehat{X}_{(T)} & & \widehat{X} \\ & {\scriptstyle h}\searrow & \downarrow & & \downarrow{\scriptstyle \hat{h}} \\ & & T & \xrightarrow[g]{} & S \end{array}$$

If instead of a bimeromorphic equivalence we only have that $X$ admits a generically finite dominant meromorphic map to $\widehat{X}_{(T)}$, then we say that $h$ *almost descends* to $S$.

This specialises the notion of descent for types introduced in Definition 3.4.

**Lemma 4.5.** *Suppose $X = \operatorname{loc}(a)$, $h : X \to T$ is a fibration and $g : T \to S$ is a dominant meromorphic map. The following are equivalent:*

(i) *$h$ almost descends to $S$,*
(ii) *$\operatorname{tp}\big(a/h(a)\big)$ almost descends to $g\big(h(a)\big)$.*

*Proof.* Let $b := h(a)$ and $b_0 := g(b) \in \operatorname{dcl}(b)$. If $\operatorname{tp}(a/b)$ almost descends to $b_0$, then there is $a_0 \in \operatorname{dcl}(a)$ such that $a \in \operatorname{acl}(ba_0)$, $a_0 \mathop{\smile\!\!\!\!\!\!\mid\;}_{b_0} b$, and $\operatorname{tp}(a_0/b_0)$ is stationary. Let $\widehat{X} := \operatorname{loc}(a_0, b_0)$ and $\hat{h} : \widehat{X} \to S$ be the second co-ordinate projection. By stationarity, $h$ is a fibration, and by independence, $\widehat{X}_{(T)}$ is the locus of $\operatorname{tp}(a_0, b)$. The fact that $a_0 \in \operatorname{dcl}(a)$ and $a \in \operatorname{acl}(ba_0)$ implies therefore, that there is a generically finite meromorphic map from $X$ to $\widehat{X}_{(T)}$ over $T$, as desired.

For the converse, assume that $h$ almost descends to $S$ witnessed by $\hat{h} : \widehat{X} \to S$ and a generically finite surjective map $f : X \to \widehat{X}_{(T)}$ over $T$. Let $a_0 \in \widehat{X}_{b_0}$ be such that $f(a) = (a_0, b)$. Note that $a_0$ is then independent from $b$ over $b_0$, and $\mathrm{tp}(a_0/b_0)$ is stationary by Fact 4.1. Now $f$ restricts to a finite-to-one surjective $b$-definable map from the realisations of $\mathrm{tp}(a/b)$ to that of $\mathrm{tp}(a_0/b)$, as desired. □

Using this lemma, Proposition 3.6 readily specialises to:

**Proposition 4.6.** *Suppose $X$ is a non-Moishezon, irreducible, compact complex space. Then the following are equivalent:*

(i) *Whenever $X \to Y$ is a dominant meromorphic map, either $Y$ is Moishezon or $\dim Y = \dim X$.*
(ii) *The algebraic reduction map $X \to B$ is minimal and does not almost descend to any $S$ with $\dim S < \dim B$.*

The following corollary thus recovers most of what is done in §2 of [1], but with some additional uniformity and the (seemingly) new observation about descent.

**Corollary 4.7.** *If $X$ is a nonalgebraic generalised hyperkähler manifold then the algebraic reduction $X \to B$ is minimal and does not almost descend to any $S$ with $\dim S < \dim B$. Moreover, either the general fibre of $X \to B$ is an abelian variety or it is uniformly isotypically semi-simple of algebraic dimension* 0 *(that is,* (II) *of Proposition 4.2 holds).*

*Proof.* By 2.3(1) of [1], $X$ satisfies condition (i) of Proposition 4.6, and hence also condition (ii). The moreover clause is by Propositions 4.2 and 4.3. □

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