THE DEGREE OF NONMINIMALITY IS AT MOST 2

JAMES FREITAG, RÉMI JAOUI, AND RAHIM MOOSA

Abstract. It is shown that if \( p \in S(A) \) is a complete type of Lascar rank at least 2, in the theory of differentially closed fields of characteristic zero, then there exists a pair of realisations \( a_1, a_2 \) such that \( p \) has a nonalgebraic forking extension over \( Aa_1a_2 \). Moreover, if \( A \) is contained in the field of constants then \( p \) already has a nonalgebraic forking extension over \( Aa_1 \). The results are also formulated in a more general setting.

1. Introduction

In [4], motivated by the search for general techniques that might aid in proving strong minimality for certain algebraic differential equations, the first and third authors introduced degree of nonminimality as a measure of how many parameters are needed to witness that a type is not minimal. Working in a sufficiently saturated model of a stable theory eliminating imaginaries, here is a precise formulation:

Definition 1.1. Suppose \( p \in S(A) \) is a stationary type with \( U(p) > 1 \). The degree of nonminimality of \( p \), denoted by \( \text{nmdeg}(p) \), is the least positive integer \( d \) such that there exist realisations \( a_1, \ldots, a_d \) of \( p \) and a nonalgebraic forking extension of \( p \) over \( Aa_1, \ldots, a_d \). If \( U(p) \leq 1 \) then we set \( \text{nmdeg}(p) = 0 \) by convention.

Using an analysis of the multiple transitivity of binding group actions, it was shown in [4] that \( \text{nmdeg}(p) \leq U(p) + 1 \) in the theory of differentially closed fields of characteristic zero (DCF\(_0\)). Bounds on the degree of nonminimality have played a significant role in recent proofs of strong minimality; of the generic differential equation in [2] and of the differential equations satisfied by the Schwarz triangle functions in [1]. Based on a maturing of the techniques used in [4], and informed by the approach taken in [3] to a related problem, we give in this note a short proof of a dramatic improvement to that bound:

Theorem. Suppose \( T = \text{DCF}_0 \) and \( p \) is a complete stationary type of finite rank. Then \( \text{nmdeg}(p) \leq 2 \). Moreover, if \( p \) is over constant parameters then \( \text{nmdeg}(p) \leq 1 \).

The bound is sharp; see [4, Example 4.2] for types of nonminimality degree 2.

The argument we give for the main clause, namely that \( \text{nmdeg}(p) \leq 2 \), works equally well in \( \text{DCF}_0,m \), the theory of differentially closed fields in \( m \) commuting derivations, and in CCM, the theory of compact complex manifolds. All one needs is...
that $T$ be totally transcendental, eliminate imaginaries, eliminate the “there exists infinitely many” quantifier, and admit a $0$-definable pure algebraically closed field to which every non locally modular minimal type is nonorthogonal. In $\text{DCF}_{0,m}$ that pure algebraically closed field is the field of constants and in CCM it is the (interpretation in $U$ of the) complex field living on the projective line.

The “moreover” clause of the theorem, however, does make use of the fact that, in $\text{DCF}_0$, the binding group of a type over the constants and internal to the constants cannot be centerless.

The most general setting for the results is articulated, for the record, in Section 3.

Remark 1.2. A corollary of our theorem is a significant improvement to the main result of [2], where it was shown that generic algebraic differential equations of order $h \geq 2$ and degree at least $2(h + 2)$ are strongly minimal. The proof in [2] used that $\text{nmdeg}(p) \leq U(p) + 1$. The same proof, but using the improved bound of $\text{nmdeg}(p) \leq 2$ obtained here, allows one to replace $2(h + 2)$ by 6 in that result.

2. The proof

We work in a fixed sufficiently saturated model $U$ of a complete totally transcendental theory $T$ eliminating imaginaries and the “there exists infinitely many” quantifier, with $C$ a $0$-definable pure algebraically closed field such that every non locally modular minimal type is nonorthogonal to $C$.

Maybe the first thing to observe is that the degree of nonminimality is invariant under interalgebraicity. Here we use the following, possibly nonstandard but unambiguous, terminology:

Definition 2.1. Complete types $p, q \in S(A)$ are said to be interalgebraic if for each (equivalently some) $a \models p$ there exists $b \models q$ such that $\text{acl}(Aa) = \text{acl}(Ab)$.

That $\text{nmdeg}(p) = \text{nmdeg}(q)$ when $p$ and $q$ are interalgebraic is more or less immediate from the definitions; see for example [4, Lemma 3.1(c)].

The following consequences of $\text{nmdeg} > 1$ were observed in [4], but we include some details here for the sake of completeness:

Fact 2.2. Suppose $p \in S(A)$ is stationary of finite rank with $\text{nmdeg}(p) > 1$. Then $p$ is interalgebraic with a stationary type $q \in S(A)$ such that $q$ is $C$-internal and $q^{(2)}$ is weakly $C$-orthogonal.

Proof. Note, first of all, that

(*) if $a \models p$ and $b \in \text{acl}(Aa) \setminus \text{acl}(A)$ then $a \in \text{acl}(Ab)$.

Indeed, if $a'$ realises the nonforking extension of $p$ to $Aab$ then $tp(a'/Aa)$ is a forking extension of $p$. Since $\text{nmdeg}(p) > 1$ we must have that $a' \in \text{acl}(Aa)$, from which it follows that $a' \in \text{acl}(Ab)$, and hence $a \in \text{acl}(Ab)$.

In the finite rank setting, condition (*), which is a weak form of exchange, implies that either $p$ is interalgebraic with a locally modular minimal type, or $p$ is almost internal to a non locally modular minimal type – see [6, Proposition 2.3]. The former is impossible as $U(p) > 1$, and by assumption on $T$ the latter implies $p$ is almost $C$-internal. We thus find a stationary $C$-internal $q \in S(A)$ that is interalgebraic with $p$. Note that $\text{nmdeg}(q) > 1$ as well.

Suppose that $q$ is not weakly $C$-orthogonal. Since the induced structure on $C$, namely that of a pure algebraically closed field, eliminates imaginaries, this failure
of weak $\mathcal{C}$-orthogonality will be witnessed by some $b \models q$ and $c \in \mathcal{C}$ such that $c \in \text{dcl}(Ab) \setminus \text{acl}(A)$. By $(*)$ applied to $q$ this would force $b \in \text{acl}(Ac)$, contradicting $U(q) > 1$. So $q$ is weakly $\mathcal{C}$-orthogonal. In particular, as it is $\mathcal{C}$-internal, $q$ is isolated. We let $\Omega$ be the definable set $q(\Omega)$.

Now suppose that $q^{(2)}$ is not weakly $\mathcal{C}$-orthogonal. Then there are independent $b_1, b_2$ realising $q$ and $c \in \mathcal{C}$ such that $c \in \text{dcl}(Ab_1b_2) \setminus \text{acl}(A)$. Note that $b_2 \notin \text{acl}(Ab_1c)$ as $U(b_2/Ab_1) = U(q) > 1$. So there is a partial $Ab_1$-definable function $f : \Omega \rightarrow \mathcal{C}$ with infinite image and infinite generic fibre. It follows, by elimination of the “there exists infinitely many” quantifier, that all but finitely many of the fibres are infinite. As $\mathcal{C} \cap \text{acl}(A)$ is infinite (it is an algebraically closed subfield of $\mathcal{C}$), there exists $b \in \Omega \setminus \text{acl}(Ab_1)$ such that $f(b) \in \text{acl}(A)$. If $b \not\in A b_1$ then $\text{tp}(b/Ab_1) = \text{tp}(b_2/Ab_1)$ contradicting the fact that $f(b_2) = c \notin \text{acl}(A)$. So $b \not\in A b_1$. That is, $\text{tp}(b/Ab_1)$ is a nonalgebraic forking extension of $q$. But this contradicts $\text{nmdeg}(q) > 1$. Hence $q^{(2)}$ is weakly $\mathcal{C}$-orthogonal.

The following improvement to Fact 2.2 was not remarked in [4].

**Lemma 2.3.** Suppose $p \in S(A)$ is stationary of finite rank with $\text{nmdeg}(p) > 1$. Then $p$ is interalgebraic with some stationary $q \in S(A)$ such that

(a) $q$ is $\mathcal{C}$-internal,

(b) $q^{(2)}$ is weakly $\mathcal{C}$-orthogonal, and,

(c) any two distinct realisations of $q$ are independent over $A$.

**Proof.** Suppose $a, b$ are realisations of $p$ such that $a \not\in A b$. If $a \notin \text{acl}(Ab)$ then $\text{tp}(a/Ab)$ is a nonalgebraic forking extension of $p$, contradicting $\text{nmdeg}(p) > 1$. Similarly, we must have $b \in \text{acl}(Ab)$. In other words, $a \not\in A b$ if and only if $\text{acl}(Ab) = \text{acl}(Ab)$. In particular, $a \not\in A b$ is an equivalence relation on $p(\Omega)$, which we now denote by $E$.

Applying Fact 2.2, we may assume that $p$ is $\mathcal{C}$-internal and $p^{(2)}$ is weakly $\mathcal{C}$-orthogonal. In particular, both $p$ and $p^{(2)}$ are isolated, say by the $L_A$-formulae $\phi(x)$ and $\psi(x, y)$, respectively. Note then, that $\phi(x) \land \phi(y) \land \neg\psi(x, y)$ defines the forking relation $E$. So $E$ is an $A$-definable equivalence relation.

Each class of $E$ is finite. Indeed, if $a \models p$ has an infinite $E$-class then there is $b \in p(\Omega) \setminus \text{acl}(Ab)$ with $aEb$. But that means that $\text{tp}(b/Ab)$ is a nonalgebraic forking extension of $p$, contradicting $\text{nmdeg}(p) > 1$.

Fixing $a \models p$, let $e := a/E$ and $q := \text{tp}(e/A)$. Note that $e \in \text{dcl}(Ab)$, and so we still have that $q$ is $\mathcal{C}$-internal and $q^{(2)}$ is weakly $\mathcal{C}$-orthogonal. Also, as the $E$-classes are finite, $p$ and $q$ are interalgebraic. So it remains to show that any two distinct realisations of $q$ are independent. Suppose $e' \models q$ with $e' \neq e$. Then $e' = a'/E$ for some $a' \models p$ such that $\neg(a Eb')$. That is $a \not\in A a'$. As $\text{acl}(Ab) = \text{acl}(Ac)$ and $\text{acl}(Ab) = \text{acl}(Ab')$, we have that $e \not\in A e'$, as desired.

We now work toward a proof of the main clause of the Theorem. That is, fixing a finite rank stationary type $p \in S(A)$, we wish to show that $\text{nmdeg}(p) \leq 2$. Let $\overline{p}$ denote the unique extension of $p$ to $\text{acl}(A)$. It is immediate from the definition that $\text{nmdeg}(\overline{p}) = \text{nmdeg}(p)$. We may therefore assume that $A = \text{acl}(A)$. Let $k := A \cap C$, it is an algebraically closed subfield of $\mathcal{C}$.

In order to prove that $\text{nmdeg}(p) \leq 2$ we may of course assume that $\text{nmdeg}(p) > 1$. Hence, by Lemma 2.3, we can further reduce to the case that $p$ is $\mathcal{C}$-internal, $p^{(2)}$ is weakly $\mathcal{C}$-orthogonal, and any two distinct realisations of $p$ are independent over $A$. 

Let \( \Omega := p(\mathcal{U}) \) and let \( G := \text{Aut}(p/\mathcal{C}) \) be the binding group of \( p \) relative to \( \mathcal{C} \). So \((G, \Omega)\) is an \( A\)-definable faithful group action. The action is transitive because \( p \) is weakly \( C\)-orthogonal. Weak \( C\)-orthogonality of \( p \) also implies, along with \( A = \text{acl}(A) \), that \( G \) is connected. The fact that \( p^{(2)} \) is weakly \( C\)-orthogonal implies that \( G \) acts transitively on \( p^{(2)}(\mathcal{U}) \). But \( p^{(2)}(\mathcal{U}) = \Omega^2 \setminus \Delta \) where \( \Delta \) is the diagonal, because any two distinct realizations of \( p \) are independent over \( A \). So \((G, \Omega)\) is a 2-transitive connected \( A\)-definable homogeneous space.

Now, the binding group action of any \( C\)-internal type is isomorphic to the \( C\)-points of an algebraic group action, though possibly over additional parameters. More precisely, let \( M \preceq \mathcal{U} \) be a prime model over \( A \). Note that \( M \cap \mathcal{C} = k \). There exists an algebraic homogeneous space \((\overline{G}, \overline{\Omega})\) defined over \( k \), and an \( M\)-definable isomorphism \( \alpha : (G, \Omega) \to (\overline{G}, \overline{\Omega}) \).

In particular, \((\overline{G}, \overline{\Omega})\) is a 2-transitive connected algebraic homogeneous space. This is a very restrictive condition; a theorem of Knop [5] tells us that \((\overline{G}, \overline{\Omega})\) is either isomorphic to the action of \( \text{PGL}_{n+1} \) on \( \mathbb{P}^n \), or is isomorphic to the action of an algebraic subgroup of the group of affine transformations on \( \mathbb{A}^n \), for some \( n > 1 \). In either case we have a notion of collinearity which is preserved by the group action. That is, given distinct \( u, v \in \overline{\Omega}(\mathcal{C}) \) we can talk about the line \( \ell_{u, v} \subseteq \overline{\Omega}(\mathcal{C}) \) connecting \( u \) and \( v \), and for all \( g \in \overline{G}(\mathcal{C}) \) we have that \( g\ell_{u, v} = \ell_{gu, gv} \).

Fix distinct \( a, b \in \Omega \), and consider the set \( X := \alpha^{-1}(\ell_{\alpha(a), \alpha(b)}) \). Then \( X \) is a rank 1 \( Mab\)-definable subset of \( \Omega \).

**Claim 2.4.** There is a finite tuple \( c \) from \( \mathcal{C} \) such that \( X \) is \( Aabc\)-definable.

**Proof.** It suffices to show that if \( \sigma \in \text{Aut}_{Aab}(\mathcal{U}/\mathcal{C}) \), that is, if \( \sigma \) is an automorphism of \( \mathcal{U} \) that fixes \( A \cup \{a, b\} \cup \mathcal{C} \) point-wise, then \( \sigma(X) = X \). Now, the restriction of \( \sigma \) to \( \Omega \) is an element of the binding group, say \( g_\sigma \in G \), which fixes \( a \) and \( b \). Hence \( \alpha(g_\sigma) \in \overline{G}(\mathcal{C}) \) fixes \( \alpha(a) \) and \( \alpha(b) \), and hence preserves the line \( \ell_{\alpha(a), \alpha(b)} \). It follows that

\[
\alpha(\sigma(X)) = \alpha(g_\sigma(\alpha^{-1}(\ell_{\alpha(a), \alpha(b)}))) = \alpha(g_\sigma)(\ell_{\alpha(a), \alpha(b)}) = \ell_{\alpha(a), \alpha(b)}.
\]

Applying \( \alpha^{-1} \) to both sides we obtain that \( \sigma(X) = X \), as desired. \( \square \)

Let \( \theta(x, y) \) be an \( L_{Aab}\)-formula such that \( X = \theta(\mathcal{U}, c) \). If, in addition, we chose \( a, b \in \Omega(M) \), then \( X \) and \( \theta(x, y) \) are over \( M \), and it follows that there is \( c' \in M \cap \mathcal{C} \) such that \( X = \theta(\mathcal{U}, c') \). But \( M \cap \mathcal{C} = k \subseteq A \), so that this witnesses the definability of \( X \) over \( Aab \).

We have thus found \( a, b \in \Omega \) and an \( Aab\)-definable subset \( X \subseteq \Omega \) of rank 1. Since \( U(p) > 1 \), the generic type of \( X \) over \( Aab \) is a nonalgebraic forking extension of \( p \). Since \( a \) and \( b \) realise \( p \), this witnesses that \( \text{nmdeg}(p) = 2 \).

This completes the proof of the main clause of the Theorem.

For the “moreover” clause, we return to the particular setting of \( T = \text{DCF}_0 \) and \( \mathcal{C} \) the field of constants. We make the additional assumption that \( A \subseteq \mathcal{C} \) and show that \( \text{nmdeg}(p) > 1 \) leads to a contradiction. Indeed, that \((G, \Omega)\) is 2-transitive forces \( G \) to be centerless; see for example the elementary argument at the beginning of the proof of Satz 2 in [5]. But, in \( \text{DCF}_0 \), the binding group of a type that is \( \mathcal{C}\)-internal and over constant parameters cannot be centerless; see for example the proof of Theorem 3.9 in [3]. This contradiction proves that \( \text{nmdeg}(p) \leq 1 \). \( \square \)
3. Some remarks on the assumptions

We carried out the above proof under assumptions on $T$ that were suitable for generalisation to both $\text{DCF}_{0,m}$ and CCM. But it may be worth recording the minimal hypotheses on $T$ required for the proofs to go through. We leave it to the reader to inspect those proofs and verify that what is actually proved are the following two statements:

**Theorem 3.1.** Suppose $T$ is a complete totally transcendental theory eliminating imaginaries and the “there exists infinitely many” quantifier. Let $U \models T$ be a sufficiently saturated model and $A \subseteq U$ a parameter set.

(a) Suppose each non locally modular minimal type in $T$ is nonorthogonal to some $A$-definable pure algebraically closed field. Then $\text{nmdeg}(p) \leq 2$ for all stationary $p \in S(A)$ of finite rank.

(b) Suppose there exists a collection $\{C_i : i \in I\}$ of $A$-definable non locally modular strongly minimal sets such that each non locally modular minimal type in $T$ is nonorthogonal to $C_i$ for some $i \in I$, and such that for all $i \in I$,

(i) $C_i \cap \text{acl}(A)$ is infinite, and,

(ii) for all $C_i$-internal $q \in S(A)$, the binding group $\text{Aut}(q/C_i)$ has a non-trivial center.

Then $\text{nmdeg}(p) \leq 1$ for all stationary $p \in S(A)$ of finite rank.

**References**


James Freitag, University of Illinois Chicago, Department of Mathematics, Statistics, and Computer Science, 851 S. Morgan Street, Chicago, IL, 60607-7045, USA

Email address: jfreitag@uic.edu

Rémi Jaoui, Albert-Ludwigs Universität Freiburg, Abteilung für Mathematische Logik, Mathematisches Institut, Ernst-Zermelo-Straße 1, D-79104 Freiburg, Germany.

Email address: remi.jaoui@math.uni-freiburg.de

Rahim Moosa, University of Waterloo, Department of Pure Mathematics, 200 University Avenue West, Waterloo, Ontario N2L 3G1, Canada

Email address: rmoosa@uwaterloo.ca