GEOMETRIC AXIOMS FOR DIFFERENTIALLY CLOSED FIELDS WITH SEVERAL COMMUTING DERIVATIONS

Omar León Sánchez
University of Waterloo, ON, Canada
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Abstract. A geometric first-order axiomatization of differentially closed fields of characteristic zero with several commuting derivations, in the spirit of Pierce-Pillay [13], is formulated in terms of a relative notion of prolongation for Kolchin-closed sets.

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1. Introduction

An ordinary differential field is a field of characteristic zero equipped with a derivation, that is, an additive map $\delta : K \to K$ such that $\delta(ab) = (\delta a)b + a(\delta b)$. A differentially closed field is a differential field (K, δ) such that every system of differential polynomial equations in several variables, with a solution in some differential extension, has a solution in K. An elegant first-order axiomatization of the class of ordinary differentially closed fields was given by Blum in [1]. In [13], Pierce and Pillay give a geometric axiomatization. Their axioms say that (K, δ) is differentially closed if and only if K is algebraically closed and whenever V and W are irreducible (affine) algebraic varieties with W contained in the prolongation of V and projecting dominantly onto V, then there is a K-point in W of the form $(\bar{x}, \delta\bar{x})$.

Similarly, a field K of characteristic zero equipped with m commuting derivations is differentially closed if every system of partial differential polynomial equations in several variables with a solution in some extension has a solution in K. A first-order axiomatization generalizing Blum's was given by McGrail in [10] (other work along these lines can be found in Tressl [15] and Yaffe [17]). However, the Pierce-Pillay condition mentioned above is no longer true for differentially closed fields with m commuting derivations (see [12], Counterexample 3.2). Nonetheless, in [12], Pierce does find first-order conditions on a subvariety W of the r-th prolongation of affine space that will ensure it has a K-point of the form $(\delta_m^{r_1} \cdots \delta_1^{r_1}\bar{x} : r_1 + \cdots + r_m \leq r)$. His criterion includes a combinatorial constraint on the algebraic relations of the coordinate functions on W. While this does lead to a geometric axiomatization of differentially closed fields, the axioms do not formally specialize to the Pierce-Pillay axioms and ultimately have a rather different flavor.

In this paper we take a very different approach, establishing an axiomatization of differentially closed fields with $(m+1)$ commuting derivations which is geometric relative to the theory with m derivations. Our axioms are a precise generalization of the Pierce-Pillay axioms, and can be used in very much the same way. Two complications arise in our setting that do not appear in the ordinary case: one has to do with extending commuting derivations and the other has to do with first-order axiomatizability. Differential-algebraic results due to Kolchin are behind our solutions to both of these problems.

Suppose $\Delta = \{\delta_1, \dots, \delta_m\}$ are commuting derivations on a field K of characteristic zero and $D : K \to K$ is an additional derivation on K that commutes with Δ. If V is a Δ-closed set defined over the D-constants of K, then Kolchin constructs a Δ-tangent bundle of V which has $\bar{x} \to (\bar{x}, D\bar{x})$ as a section ([6], Chap. VIII, §2). In general, if V is not necessarily defined over the D-constants, then D gives a section of a certain torsor of the Δ-tangent bundle of V that we call the *D/Δ-prolongation* of V (cf. Definition 3.1). Our axioms will essentially say that $(K, \Delta \cup \{D\})$ *is differentially closed if and only if K is algebraically closed and whenever V and W are Δ-closed sets with W contained in the D/Δ-prolongation of V and projecting onto V, then there is a K-point in W of the form $(\bar{x}, D\bar{x})$*. "Essentially", because in actual fact we also have to consider not just Δ and D but also various linear combinations of them (cf. Theorem 4.3 below).

Pierce-Pillay type axiomatizations have been obtained in various other contexts: difference fields (Chatzidakis and Hrushovski [3]), difference-differential fields (Bustamante [2]), derivations of the Frobenius and commuting Hasse-Schmidt derivations in positive characteristic (Kowalski [7], [8]). However, the techniques used in these works do not seem to translate to our context.

The paper is organized as follows. In Section 2 we establish the differential-algebraic facts that underpin our results. In Section 3 we introduce relative prolongations and prove a geometric characterization of differentially closed fields. Finally, in Section 4, we address the issue of first-order axiomatizability.

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2. Extending Δ-derivations

In this paper the term ring is used for commutative ring with unity and the term field for field of characteristic zero.

Let us first recall some terminology from differential algebra. For details see [5]. Let R be a ring and S a ring extension. An additive map $\delta : R \to S$ is called a derivation if it satifies the Leibniz rule; i.e., $\delta(ab) = (\delta a)b + a(\delta b)$. A ring R equipped with a set of derivations $\Delta = \{\delta_1, \dots, \delta_m\}$, $\delta_i : R \to R$, such that the derivations commute with each other is called a Δ-ring. A Δ-ring which is also a field (of characteristic zero) is called a Δ-field.

We fix for the rest of this section a Δ-ring R. Let Θ denote the free commutative monoid generated by Δ; that is,

$$\Theta := \{\delta_m^{r_m} \cdots \delta_1^{r_1} : r_m, \dots, r_1 \geq 0\}.$$

The elements of Θ are called the derivative operators. Let $\bar{x} = (x_1, \dots, x_n)$ be a family of indeterminates, and define

$$\theta\bar{x} := \{\partial x_j : j = 1, \dots, n, \, \partial \in \Theta\}.$$

The Δ-ring of Δ-polynomials over R in the differential indeterminates $\bar{x}$ is $R\{\bar{x}\} := R[\theta\bar{x}]$; that is, the ring of polynomials in the algebraic indeterminates $\theta\bar{x}$ with the canonical Δ-ring structure given by $\delta_i(\delta_m^{r_m} \cdots \delta_1^{r_1} x_j) = \delta_m^{r_m} \cdots \delta_i^{r_i+1} \cdots \delta_1^{r_1} x_j$.

We fix an orderly ranking in $\theta\bar{x}$ by:

$$\delta_m^{r_m}\cdots\delta_1^{r_1}x_i \leq \delta_m^{r'_m}\cdots\delta_1^{r'_1}x_j \iff \left(\sum r_l, i, r_m, \dots, r_1\right) \leq \left(\sum r'_l, j, r'_m, \dots, r'_1\right)$$

in the lexicographical order. According to this ranking, we enumerate the algebraic indeterminates by $\theta\bar{x} = (\theta_1\bar{x}, \theta_2\bar{x}, \dots)$. Therefore, if $f \in R\{\bar{x}\}$ there is a unique $\hat{f} \in R[t_1, t_2, \dots]$ such that $f(\bar{x}) = \hat{f}(\theta\bar{x})$.

We will be interested in adding an extra derivation on R.

Definition 2.1. Let S be a Δ-ring extension of R. A Δ-*derivation* from R to S is a derivation $D : R \to S$ such that $D\delta = \delta D$ for all $\delta \in \Delta$.

Fix a Δ-ring extension S of R and a Δ-derivation $D : R \to S$. We are interested in the extensions of D to Δ-derivations from finitely generated Δ-ring extensions of R to S. This subject was studied by Kolchin in ([6], Chapter 0, §4). We will need the following terminology to present the main results. If $f \in R\{\bar{x}\}$, by f^D we mean the Δ-polynomial in $S\{\bar{x}\}$ obtained by applying D to the coefficients of f. Note that the map $f \mapsto f^D$ is itself a Δ-derivation from $R\{\bar{x}\}$ to $S\{\bar{x}\}$. By the *Jacobian* of f we will mean

$$df(\bar{x}) := \left(\frac{\partial \hat{f}}{\partial t_i}(\theta\bar{x})\right)_{i\in\mathbb{N}}$$

viewed as an element of $(R\{\bar{x}\})^{\mathbb{N}}$. Note that df is finitely supported, in the sense that all but finitely many coordinates are zero.

Remark 2.2. Suppose $\bar{a}$ is a tuple of S and $D' : R\{\bar{a}\} \to S$ is a Δ-derivation extending D. If $f \in R\{\bar{x}\}$, then an easy computation shows that

$$D'f(\bar{a}) = df(\bar{a}) \cdot \theta D'\bar{a} + f^D(\bar{a}).$$

Here if $\bar{a} = (a_1, \dots, a_n)$ then $D'\bar{a} = (D'a_1, \dots, D'a_n)$ and $\theta D'\bar{a} = (\theta_1 D'\bar{a}, \theta_2 D'\bar{a}, \dots)$. Note that the dot product is well defined since df has finite support.

Definition 2.3. Let $f \in R\{\bar{x}\}$. We define the Δ-polynomial $\tau_{D/\Delta}f \in S\{\bar{x}, \bar{y}\}$ by

$$\tau_{D/\Delta}f(\bar{x}, \bar{y}) := df(\bar{x}) \cdot \theta\bar{y} + f^D(\bar{x}).$$

When Δ and D are understood we simply write τf. If $\bar{a} \in S$, we write $\tau(f)_{\bar{a}}(\bar{y})$ for $\tau f(\bar{a}, \bar{y}) \in S\{\bar{y}\}$. Note that $\tau\theta\bar{x} = \theta\bar{y}$ and if $c \in R$ then $\tau c = Dc$.

Note that, under the assumptions of Remark 2.2, for all f in the prime Δ-ideal $\mathcal{I}(\bar{a}/R) := \{f \in R\{\bar{x}\} : f(\bar{a}) = 0\}$ we get

$$\tau(f)_{\bar{a}}(D'\bar{a}) = D'f(\bar{a}) = 0.$$

Thus any Δ-derivation D' from $R\{\bar{a}\}$ to S extending D gives a tuple $D'\bar{a}$ of S at which $\tau(f)_{\bar{a}}$ vanishes for all $f \in \mathcal{I}(\bar{a}/R)$. The following proposition is the converse of this implication and gives a criterion for when a Δ-derivation can be extended to a finitely generated Δ-ring extension. The case when $\Delta = \emptyset$ can be found in ([9], Chap. 7, §5), and is the main point in the Pierce-Pillay geometric axiomatization of ordinary differentially closed fields.

Proposition 2.4 ([6], Chap. 0, §4)**.** *Let $D : R \to S$ be a Δ-derivation and $\bar{a}$ a tuple of S. Suppose there is a tuple $\bar{b}$ of S such that*

$$\tau(f)_{\bar{a}}(\bar{b}) = 0, \text{ for all } f \in \mathcal{I}(\bar{a}/R). \tag{2.1}$$

Then there is a unique Δ-derivation $D' : R\{\bar{a}\} \to S$ extending D such that $D'\bar{a} = \bar{b}$.

Thus if we want to extend D to a Δ-derivation from $R\{\bar{a}\}$ to S, we need to find a solution of the system of Δ-equations $\{\tau(f)_{\bar{a}}(\bar{y}) = 0 : f \in \mathcal{I}(\bar{a}/R)\}$. In the case when S is a field, Kolchin showed that this system does have a solution in some Δ-field extension of S. Indeed he shows ([6], Chap. 0, §4, Proposition 5) that the ideal generated by $\{\tau(f)_{\bar{a}}(\bar{y}) : f \in \mathcal{I}(\bar{a}/R)\}$ in $S\{\bar{y}\}$ is a prime Δ-ideal. From this and Proposition 2.4 one obtains:

Corollary 2.5 ([6], Chap. 0, §4)**.** *Suppose (K, Δ) is a differentially closed field extending R and $D : R \to K$ a Δ-derivation. Then there is a Δ-derivation $D' : K \to K$ extending D.*

We will require an improvement on Proposition 2.4. We would like to only have to check condition (2.1) for a set of Δ-polynomials $A \subset R\{\bar{x}\}$ such that $\{A\} = \mathcal{I}(\bar{a}/R)$, where $\{A\}$ denotes the radical Δ-ideal generated by A. As the reader may expect this will be useful when dealing with issues of first-order axiomatizability (see Proposition 3.2 below).

First we need a lemma. For each $i = 1, 2, \dots$, let $\bar{x}_i$ be an n-tuple of differential indeterminates. Suppose $D : R \to R$ is a Δ-derivation. Then $\tau : R\{\bar{x}_1\} \to R\{\bar{x}_1, \bar{x}_2\}$. Thus we can compose τ with itself, for each $k \geq 1$ and $f \in R\{\bar{x}_1\}$, $\tau^k f = \tau \cdots \tau f \in R\{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{2^k}\}$. Define $\nabla\bar{x} := (\bar{x}, D\bar{x})$ and note that, for each $k \geq 1$, the composition $\nabla^k \bar{x} = \nabla \cdots \nabla \bar{x}$ is a tuple of length $n2^k$.

Lemma 2.6. *Suppose $D : R \to R$ is a Δ-derivation and $f \in R\{\bar{x}_1\}$.*

(1) *If $\bar{a}$ is a tuple of R, then for each $k \geq 1$,*

$$\tau^k f(\nabla^k \bar{a}) = D^k f(\bar{a})$$

In particular, if $f(\bar{a}) = 0$ then $\tau^k f(\nabla^k \bar{a}) = 0$.

(2) *For each $k \geq 1$, we have*

$$\tau^k f^k = k!(\tau f)^k + f\, p$$

for some $p \in R\{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{2^k}\}$.

Proof. (1) By induction on k. Remark 2.2 gives us

$$\tau f(\nabla \bar{a}) = df(\bar{a}) \cdot \theta D\bar{a} + f^D(\bar{a}) = Df(\bar{a}).$$

The induction step follows easily:

$$\tau^{k+1} f(\nabla^{k+1}\bar{a}) = \tau(\tau^k f)(\nabla(\nabla^k \bar{a})) = D\tau^k f(\nabla^k \bar{a}) = DD^k f(\bar{a}) = D^{k+1} f(\bar{a}).$$

(2) We prove that for each $l = 1, \dots, k$ we have

$$\tau^l(f^k) = \frac{k!}{(k-l)!} f^{k-l}(\tau f)^l + f^{k-l+1}\, p_l \tag{2.2}$$

for some $p_l \in K\{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{2^l}\}$. From which the results follows when $l = k$. Since $\tau f^k = k f^{k-1} \tau f$, we get (2.2) holds for $l = 1$ with $p_1 = 0$. Assume it holds for

$1 \leq l < k$, then

$$\begin{aligned}
\tau^{l+1} f^k &= \tau\tau^l f^k = \tau\left(\frac{k!}{(k-l)!} f^{k-l}(\tau f)^l + f^{k-l+1} p_l\right) \\
&= \frac{k!}{(k-l)!}\left((k-l) f^{k-l-1}(\tau f)^{l+1} + l f^{k-l}(\tau f)^{l-1}\tau^2 f\right) \\
&\quad + (k-l+1) f^{k-l}(\tau f)\, p_l + f^{k-l+1}\tau p_l \\
&= \frac{k!}{(k-l-1)!} f^{k-l-1}(\tau f)^{l+1} + f^{k-l}\, p_{l+1}
\end{aligned}$$

where

$$p_{l+1} = \frac{k!\, l}{(k-l)!}(\tau f)^{l-1}\tau^2 f + (k-l+1)(\tau f)\, p_l + f\tau p_l.$$

$\square$

Proposition 2.7. *Suppose R is a reduced $\mathbb{Q}$-algebra and $D : R \to R$ is a Δ-derivation. Let $\bar{a}$ be a tuple of R and $A \subseteq \mathcal{I}(\bar{a}/R)$. Suppose there is a tuple $\bar{b}$ of R such that*

$$\tau(f)_{\bar{a}}(\bar{b}) = 0, \ \text{for all } f \in A. \tag{2.3}$$

Then $\tau(f)_{\bar{a}}(\bar{b}) = 0$ for all $f \in \{A\}$.

Proof. First we show equation (2.3) holds for all f in $[A]$, where $[A]$ is the Δ-ideal generated by A. For each $\partial \in \Theta$, $f \in A$ and $h \in R\{\bar{x}\}$, we have

$$\tau(h\partial f)_{\bar{a}}(\bar{b}) = \tau(h)_{\bar{a}}(\bar{b})\partial f(\bar{a}) + h(\bar{a})\partial(\tau(f)_{\bar{a}}(\bar{b})). \tag{2.4}$$

Here we used the fact that $\tau(\partial f)_{\bar{a}}(\bar{b}) = \partial(\tau(f)_{\bar{a}}(\bar{b}))$ (see [6], Chap. 0, §4, pp.9). By assumption $\tau(f)_{\bar{a}}(\bar{b}) = 0$ and since $f \in A \subseteq \mathcal{I}(\bar{a}/R)$ we get $\partial f(\bar{a}) = 0$. Thus (2.4) yields $\tau(h\partial f)_{\bar{a}}(\bar{b}) = 0$. It follows that for each $f \in [A]$, $\tau(f)_{\bar{a}}(\bar{b}) = 0$.

Now let $f \in \{A\}$, since $R\{\bar{x}\}$ is also a $\mathbb{Q}$-algebra $\{A\} = \sqrt{[A]}$, and so there is $k \geq 1$ such that $f^k \in [A]$ and hence $\tau f^k(\bar{a}, \bar{b}) = 0$. By part (1) of Lemma 2.6,

$$\tau^k f^k(\nabla^{k-1}(\bar{a}, \bar{b})) = \tau^{k-1}(\tau f^k)(\nabla^{k-1}(\bar{a}, \bar{b})) = D^{k-1}\tau f^k(\bar{a}, \bar{b}) = 0.$$

Thus, by part (2) of Lemma 2.6, we have

$$k!(\tau f(\bar{a}, \bar{b}))^k + f(\bar{a}) p(\nabla^{k-1}(\bar{a}, \bar{b})) = 0,$$

for some $p \in R\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{2^k}\}$. Since $f(\bar{a}) = 0$, we get $k!(\tau f(\bar{a}, \bar{b}))^k = 0$. Thus, since R is a reduced $\mathbb{Q}$-algebra, $\tau(f)_{\bar{a}}(\bar{b}) = \tau f(\bar{a}, \bar{b}) = 0$. $\square$

Corollary 2.8. *If S is a field, then Proposition 2.4 holds even if we replace $\mathcal{I}(\bar{a}/R)$ for any $A \subset R\{\bar{x}\}$ such that $\{A\} = \mathcal{I}(\bar{a}/R)$.*

Proof. Suppose $\{A\} = \mathcal{I}(\bar{a}/R)$ and $\tau_{D/\Delta}(f)_{\bar{a}}(\bar{b}) = 0$ for all $f \in A$. We need to show that $\tau_{D/\Delta}(f)_{\bar{a}}(\bar{b}) = 0$ for all $f \in \mathcal{I}(\bar{a}/R)$. Let (K, Δ) be a differentially closed field extending S. By Corollary 2.5, we can extend D to a derivation $D' : K \to K$. Now, by Proposition 2.7, $\tau_{D'/\Delta}(f)_{\bar{a}}(\bar{b}) = 0$ holds for all $f \in \{A\}_K$, where $\{A\}_K$ denotes the radical Δ-ideal in $K\{\bar{x}\}$ generated by A. But $\{A\} \subseteq \{A\}_K$, so that $\tau_{D/\Delta}(f)_{\bar{a}}(\bar{b}) = 0$ for all $f \in \mathcal{I}(\bar{a}/R)$, as desired. $\square$

3. Relative prolongations and a characterization of $DCF_{0,m+1}$

Let us recall the notion of prolongation for ordinary differential fields. Given a δ-field K and V a Zariski-closed set of K^n, the prolongation of V, τV, is the Zariski-closed subset of K^{2n} defined by the equations $f(\bar{x}) = 0$ and $\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\bar{x}) y_i + f^{\delta}(\bar{x}) = 0$, for each polynomial $f \in K[\bar{x}]$ vanishing on V. Note that, in terms of our notation from Definition 2.3, the last equation is just $\tau_{\delta/\emptyset} f(\bar{x}, \bar{y}) = 0$.

Fix a differential field $(K, \Delta \cup \{D\})$ with $\Delta = \{\delta_1, \ldots, \delta_m\}$. We introduce a prolongation for Δ-closed sets with respect to D.

Definition 3.1. Suppose $V \subseteq K^n$ is a Δ-closed set. The D/Δ-*prolongation* of V, $\tau_{D/\Delta} V \subseteq K^{2n}$, is the Δ-closed set defined by

$$f = 0 \text{ and } \tau_{D/\Delta} f = 0, \text{ for all } f \in \mathcal{I}(V/K).$$

Here $\mathcal{I}(V/K) = \{f \in K\{\bar{x}\} : f \text{ vanishes on } V\}$. When Δ and D are understood, we just write τf and τV. For $\bar{a} \in V$, $\tau(V)_{\bar{a}}$ denotes the fibre of τV at $\bar{a}$. Note that when $m = 0$ this is consistent with the ordinary case discussed above.

By Remark 2.2, if $\bar{a}$ is in V then $(\bar{a}, D\bar{a}) \in \tau V$. This implies that the projection $\pi : \tau V \to V$ given by $\pi(\bar{x}, \bar{y}) = \bar{x}$ is surjective and that $\bar{x} \mapsto (\bar{x}, D\bar{x})$ is a section.

We work in the language of differential rings, $\mathcal{L}_m = \{0, 1, +, -, \times, \delta_1, \ldots, \delta_m\}$, we denote by $DF_{0,m}$ the theory of differential fields of characteristic zero with m commuting derivations, and we denote by $DCF_{0,m}$ its model-completion, the theory of differentially closed fields. The following consequence of Proposition 2.7 says that in a model of $DCF_{0,m}$ the D/Δ-prolongation varies uniformly with V.

Proposition 3.2. *Suppose* $(K, \Delta) \models DCF_{0,m}$. *If* $V = \mathcal{V}(f_1, \ldots, f_s) := \{\bar{a} \in K^n : f_i(\bar{a}) = 0, i = 1, \ldots, s\}$, *then* $\tau V = \mathcal{V}(f_i, \tau f_i : i = 1, \ldots, s)$.

Proof. Clearly $\tau V \subseteq \mathcal{V}(f_i, \tau f_i : i = 1, \ldots, s)$. Let $(\bar{a}, \bar{b}) \in \mathcal{V}(f_i, \tau f_i : i = 1, \ldots, s)$. By Proposition 2.7, $\tau f(\bar{a}, \bar{b}) = 0$ for all $f \in \{f_1, \ldots, f_s\}$. Since $(K, \Delta) \models DCF_{0,m}$, we have $\{f_1, \ldots, f_s\} = \mathcal{I}(\mathcal{V}(f_1, \ldots, f_s)/K) = \mathcal{I}(V/K)$. Hence, $(\bar{a}, \bar{b}) \in \tau V$. □

Remark 3.3.

(1) Suppose $(K, \Delta) \models DCF_{0,m}$. If V is defined over the D-constants, that is, $V = \mathcal{V}(f_1, \ldots, f_s)$ where $f_i \in \mathcal{C}_D\{\bar{x}\}$, then τV is just Kolchin's Δ-tangent bundle of V. Indeed, by Proposition 3.2, the equations defining τV become $f_i(\bar{x}) = 0$ and $\tau f_i(\bar{x}, \bar{y}) = df_i(\bar{x}) \cdot \theta \bar{y} = 0$, $i = 1, \ldots, s$. These are exactly the equations for Kolchin's Δ-tangent bundle $T_\Delta V$ ([6], Chap.VIII, §2).

(2) In general, τV is a torsor under $T_\Delta V$. Indeed, from the equations one sees that $\tau(V)_{\bar{a}}$ is a translate of $T_\Delta(V)_{\bar{a}}$, and so the map $T_\Delta V \times_V \tau V \to \tau V$ given by $((\bar{a}, \bar{b}), (\bar{a}, \bar{c})) \mapsto (\bar{a}, \bar{b} + \bar{c})$ is a regular action of $T_\Delta V$ on τV over V.

Note that in case $\Delta = \emptyset$, part (2) of Remark 3.3 reduces to the fact that the prolongation of a Zariski-closed set is a torsor under its tangent bundle.

The following characterization of $DCF_{0,m+1}$ will be used in the next section to obtain a geometric first-order axiomatization.

Theorem 3.4. *Suppose* $(K, \Delta \cup \{D\}) \models DF_{0,m+1}$. *Then* $(K, \Delta \cup \{D\}) \models DCF_{0,m+1}$ *if and only if*

(1) $(K, \Delta) \models DCF_{0,m}$

(2) *For each pair of irreducible Δ-closed sets $V \subseteq K^n$, $W \subseteq \tau V$ such that W projects Δ-dominantly onto V. If O_V and O_W are nonempty Δ-open subsets of V and W respectively, then there exists $\bar{a} \in O_V$ such that $(\bar{a}, D\bar{a}) \in O_W$.*

As we will see in the proof, it would have been equivalent in condition (2) to take $O_V = V$ and $O_W = W$. Also note that, under the convention that $DCF_{0,0}$ is the theory of algebraically closed fields of characteristic zero ACF_0, when $m = 0$ this is exactly the Pierce-Pillay axioms ([13], §2).

Proof. Suppose $(K, \Delta \cup \{D\}) \models DCF_{0,m+1}$, and V, W, O_V and O_W are as in condition (2). Let $(\mathbb{U}, \Delta)$ be a large differentially closed field; i.e., a universal domain for Δ-algebraic geometry. If X is an ($\mathcal{L}_m$-)definable subset of K^n, by $X(\mathbb{U})$ we mean the interpretation of X in $\mathbb{U}^n$. Let $(\bar{a}, \bar{b}) \in \mathbb{U}^{2n}$ be a Δ-generic point of W over K; that is, $\mathcal{I}(\bar{a}, \bar{b}/K) = \mathcal{I}(W(\mathbb{U})/K)$. Then $(\bar{a}, \bar{b}) \in O_W(\mathbb{U})$. Since $(\bar{a}, \bar{b}) \in \tau V(\mathbb{U})$ we have that $\tau(f)_{\bar{a}}(\bar{b}) = 0$ for all $f \in \mathcal{I}(V(\mathbb{U})/K)$. The fact that W projects Δ-dominantly onto V implies that $\bar{a}$ is a Δ-generic point of V over K, so $\bar{a} \in O_V(\mathbb{U})$ and $\mathcal{I}(\bar{a}/K) = \mathcal{I}(V(\mathbb{U})/K)$. Hence, $\tau(f)_{\bar{a}}(\bar{b}) = 0$ for all $f \in \mathcal{I}(\bar{a}/K)$. By Proposition 2.4, there is a unique Δ-derivation $D' : K\{\bar{a}\} \to \mathbb{U}$ extending D such that $D'\bar{a} = \bar{b}$. By Corollary 2.5, we can extend D' to all of $\mathbb{U}$, call it D''. Hence, $\mathbb{U}$ becomes a $\Delta \cup \{D''\}$-field extending the $\Delta \cup \{D\}$-closed field K. Since $\bar{a} \in O_V(\mathbb{U})$, $(\bar{a}, \bar{b}) \in O_W(\mathbb{U})$ and $D''\bar{a} = \bar{b}$, we get a point $(\bar{a}', \bar{b}')$ in K such that $\bar{a}' \in O_V$, $(\bar{a}', \bar{b}') \in O_W$ and $D\bar{a}' = \bar{b}'$.

The converse is essentially as in [13]. For the sake of completeness we give the details. Let $\phi(\bar{x})$ be a conjunction of atomic $\mathcal{L}_{m+1}$-formulas over K. Suppose ϕ has a realisation $\bar{a}$ in some $(F, \Delta \cup \{D\}) \models DF_{0,m+1}$ extending of $(K, \Delta \cup \{D\})$. Let

$$\phi(\bar{x}) = \psi(\bar{x}, \delta_{m+1}\bar{x}, \dots, \delta_{m+1}^r \bar{x})$$

where ψ is a conjunction of atomic $\mathcal{L}_m$-formulas over K and $r > 0$. Let $\bar{c} = (\bar{a}, D\bar{a}, \dots, D^{r-1}\bar{a})$ and $X \subseteq F^{nr}$ be the Δ-locus of $\bar{c}$ over K. Let $Y \subseteq F^{2nr}$ be the Δ-locus of $(\bar{c}, D\bar{c})$ over K. Let

$$\chi(\bar{x}_0, \dots, \bar{x}_{r-1}, \bar{y}_0, \dots, \bar{y}_{r-1}) := \psi(\bar{x}_0, \dots, \bar{x}_{r-1}, \bar{y}_{r-1}) \wedge \left(\wedge_{i=1}^{r-1} \bar{x}_i = \bar{y}_{i-1}\right)$$

then χ is realised by $(\bar{c}, D\bar{c})$. Since $(\bar{c}, D\bar{c})$ is a Δ-generic point of Y over K and its projection $\bar{c}$ is a Δ-generic point of X over K, we have that Y projects Δ-dominantly onto X over K. Thus, since $(K, \Delta) \models DCF_{0,m}$, $Y(K)$ projects Δ-dominantly onto $X(K)$. Also, since $(\bar{c}, D\bar{c}) \in \tau X$, we have $Y(K) \subseteq \tau(X(K))$. Applying (2) with $V = O_V = X(K)$ and $W = O_W = Y(K)$, there is $\bar{d}$ in V such that $(\bar{d}, D\bar{d}) \in W$. Let $\bar{d} = (\bar{d}_0, \dots, \bar{d}_{r-1})$ then $(\bar{d}_0, \dots, \bar{d}_{r-1}, D\bar{d}_0, \dots, D\bar{d}_{r-1})$ realises χ. Thus, $(\bar{d}_0, D\bar{d}_0, \dots, D^r\bar{d}_0)$ realises ψ. Hence, $\bar{d}_0$ is a tuple of K realising ϕ. This proves that $(K, \Delta \cup \{D\}) \models DCF_{0,m+1}$. □

4. Geometric first-order axioms

The Pierce-Pillay characterization of DCF_0, that is Theorem 3.4 when $m = 0$, is indeed first-order. Expressing irreducibility of a Zariski-closed set as a definable condition on the parameters uses the existence of bounds to check primality of ideals in polynomial rings in finitely many variables [16]. Also, if the field is algebraically closed, one can find a first-order formula, in the language of rings, describing for which parameters a Zariski-closed set projects dominantly onto some fixed irreducible Zariski-closed set. This follows from the fact that algebraic-geometric dimension is definable in ACF_0.

It is not known to the author if condition (2) of Theorem 3.4 can be expressed in a first-order way for $m > 0$. One issue is to express irreducibility of Δ-closed sets as a definable condition. This seems to be an open problem related to the generalized Ritt problem [11]. The other issue is how to express when a Δ-closed set projects Δ-dominantly onto another Δ-closed set as a definable condition. Unlike the algebraic case, in differentially closed fields, Noetherian dimension (in the Kolchin topology) is not definable ([4], §2).

We resolve this problem by modifying the characterization of Theorem 3.4 so that it no longer mentions irreducibility or dominance. The first of these can be handled rather easily by the following lemma.

Lemma 4.1. *Let K be a $\Delta \cup \{D\}$-field. Let $V \subseteq K^n$ be a Δ-closed set with K-irreducible components $\{V_1, \dots, V_s\}$. If $\bar{a} \in V_i \setminus \bigcup_{j \neq i} V_j$, then $\tau(V)_{\bar{a}} = \tau(V_i)_{\bar{a}}$.*

Proof. Clearly $\tau(V_i)_{\bar{a}} \subseteq \tau(V)_{\bar{a}}$. Let $\bar{b} \in \tau(V)_{\bar{a}}$ and $f \in \mathcal{I}(V_i/K)$. Since $\bar{a}$ is not in V_j, for $j \neq i$, we can pick a $g_j \in \mathcal{I}(V_j/K)$ such that $g_j(\bar{a}) \neq 0$. Then, if $g = \prod_j g_j$, we get $fg \in \mathcal{I}(V/K)$ and so

$$0 = \tau(fg)_{\bar{a}}(\bar{b}) = \tau(f)_{\bar{a}}(\bar{b})g(\bar{a}) + f(\bar{a})\tau(g)_{\bar{a}}(\bar{b}) = \tau(f)_{\bar{a}}(\bar{b})g(\bar{a})$$

where the third equality holds because $\bar{a} \in V_i$. Since $g(\bar{a}) \neq 0$, we have $\tau(f)_{\bar{a}}(\bar{b}) = 0$, and so $\bar{b} \in \tau(V_i)_{\bar{a}}$. □

It follows that if $W \subseteq \tau V$ projects Δ-dominantly onto V and V_i is a K-irreducible component of V, then a K-irreducible component of $W \cap \tau V_i$ projects Δ-dominantly onto V_i.

The second issue, that of Δ-dominant projections, is more difficult to deal with. Let us note here that when $\Delta = \emptyset$, that is, in the case of DCF_0, one can just replace dominant projections by surjective projections in the Pierce-Pillay axiomatization. Indeed this reformulation is stated in [14]. We will not give a proof here as it will follow from Theorem 4.3 below. However, what makes this work, in the case of a single derivation D, is the fact that if a is D-algebraic over K, then $D^{k+1}a \in K(a, Da, \dots, D^k a)$ for some k. In several derivations it is not necessarily the case that if a is $\Delta \cup \{D\}$-algebraic over K, then $D^{k+1}a$ is in the Δ-field generated by $a, Da, \dots, D^k a$ over K, for some k. However, by a theorem of Kolchin (Proposition 4.2 below), this can always be achieved if we allow $\mathbb{Z}$-linear transformations of the derivations. Our modification of Theorem 3.4 will therefore need to refer to such transformations.

For every $M = (c_{i,j}) \in \mathrm{SL}_{m+1}(\mathbb{Z})$, let $\Delta' = \{\delta_1', \dots, \delta_m'\}$ and D' be the derivations on K defined by $\delta_i' = c_{i,1}\delta_1 + \dots + c_{i,m}\delta_m + c_{i,m+1}D$ and $D' = c_{m+1,1}\delta_1 + \dots + c_{m+1,m}\delta_m + c_{m+1,m+1}D$. In this case we write $(\Delta', D') = M(\Delta, D)$. Clearly, the elements of $\Delta' \cup \{D'\}$ are also commuting derivations on K.

Proposition 4.2 ([5], Chap. II, §11)**.** *Let $(K, \Delta \cup \{D\}) \models DF_{0,m+1}$. Let $\bar{a} = (a_1, \dots, a_n)$ be a tuple of a $\Delta \cup \{D\}$-field extension of K. Suppose all the a_i's are $\Delta \cup \{D\}$-algebraic over K, then there exists $k > 0$ and a matrix $M \in SL_{m+1}(\mathbb{Z})$ such* *that, writing* $(\Delta', D') = M(\Delta, D)$, *we have that* $D'^{\ell}\bar{a}$ *is in the* Δ'*-field generated by* $\bar{a}, D'\bar{a} \dots, D'^{k}\bar{a}$ *over* K*, for all* $\ell > k$*.*

Theorem 3.4 characterizes $DCF_{0,m+1}$ in terms of the geometry of $DCF_{0,m}$. The idea, of course, was that $DCF_{0,m}$ has a similar characterization relative to $DCF_{0,m-1}$, and so on. In Theorem 4.3 we will implement this recursion and give a geometric first-order axiomatization of $DCF_{0,m+1}$ for all $m \geq 0$, that refers only to the base theory ACF_0.

Theorem 4.3. *Suppose* $(K, \Delta \cup \{D\}) \models DF_{0,m+1}$*. Then* $(K, \Delta \cup \{D\}) \models DCF_{0,m+1}$ *if and only if*

1. $K \models ACF_0$
2. *Suppose* $M \in SL_{m+1}(\mathbb{Z})$, $(\Delta', D') := M(\Delta, D)$, $V = \mathcal{V}(f_1, \dots, f_s) \subseteq K^n$ *is a nonempty* Δ'*-closed set, and*

$$W \subseteq \mathcal{V}(f_1, \dots, f_s, \tau_{D'/\Delta'} f_1, \dots, \tau_{D'/\Delta'} f_s) \subseteq K^{2n}$$

is a Δ'*-closed set that projects onto* V*. Then there is* $\bar{a} \in V$ *such that* $(\bar{a}, D'\bar{a}) \in W$*.*

Proof. Suppose $(K, \Delta \cup \{D\}) \models DCF_{0,m+1}$. Clearly $K \models ACF_0$. Suppose M, Δ', V and W are as in condition (2). Clearly $(K, \Delta' \cup \{D'\}) \models DCF_{0,m+1}$, so by Proposition 3.2 we have that $\mathcal{V}(f_i, \tau_{D'/\Delta'} f_i : i = 1, \dots, s) = \tau_{D'/\Delta'} V$. Let V_i be an irreducible component of V and $W' = W \cap \tau_{D'/\Delta'} V_i$. By Lemma 4.1, we can find an irreducible component of W' projecting Δ'-dominantly onto V_i. Now just apply Theorem 3.4 (with $\Delta' \cup \{D'\}$ rather than $\Delta \cup \{D\}$) to get the desired point.

For the converse, we assume conditions (1) and (2) and prove that $(K, \Delta \cup \{D\}) \models DCF_{0,m+1}$. Given $r = 1, \dots, m+1$ and $N \in \mathrm{SL}_{m+1}(\mathbb{Z})$, let $\mathcal{K}_{r,N} = (K, \bar{\Delta}_{r-1} \cup \{\bar{D}\})$ where $(\bar{\Delta}, \bar{D}) = N(\Delta, D)$ and $\bar{\Delta}_{r-1} = \{\bar{\delta}_1, \dots, \bar{\delta}_{r-1}\}$. Set $\mathcal{K}_{0,N}$ to be the pure algebraic field K. We show by induction that for each $r = 0, \dots, m+1$, $\mathcal{K}_{r,N} \models DCF_{0,r}$ for all $N \in \mathrm{SL}_{m+1}(\mathbb{Z})$. The result will then follow by setting $r = m+1$ and $N = \mathrm{Id}$. The case of $r = 0$ is just assumption (1). We assume $0 \leq r \leq m$, $N \in \mathrm{SL}_{m+1}(\mathbb{Z})$, and we show that $\mathcal{K}_{r+1,N} = (K, \bar{\Delta}_r \cup \{\bar{D}\}) \models DCF_{0,r+1}$.

Suppose $\phi(\bar{x})$ is a conjunction of atomic $\mathcal{L}_{r+1}$-formulas over K, with a realisation $\bar{a} = (a_1, \dots, a_n)$ in some $\bar{\Delta}_r \cup \{\bar{D}\}$-field F extending $\mathcal{K}_{r+1,N}$. We need to find a realisation of ϕ in $\mathcal{K}_{r+1,N}$. We may assume that each a_i is $\bar{\Delta}_r \cup \{\bar{D}\}$-algebraic over K (this can be seen algebraically or one can use the existence of prime models of $DCF_{0,r+1}$ over K, see §3.2 of [10]).

Let $M' \in \mathrm{SL}_{r+1}(\mathbb{Z})$ and $k > 0$ be the matrix and natural number given by Proposition 4.2. Let $M \in \mathrm{SL}_{m+1}(\mathbb{Z})$ be

$$M = E \begin{pmatrix} M' & 0 \\ 0 & I \end{pmatrix} EN$$

where E is the elementary matrix of size $(m+1)$ that interchanges row $(r+1)$ with row $(m+1)$ and I is the identity matrix of size $(m-r)$. Then, setting $(\Delta', D') = M(\Delta, D)$, we get

$$D'^{k+1}\bar{a} = \frac{f(\bar{a}, D'\bar{a}, \dots, D'^{k}\bar{a})}{g(\bar{a}, D'\bar{a} \dots, D'^{k}\bar{a})} \tag{4.1}$$

for some $f, g \in (K\{\bar{x}_0, \ldots, \bar{x}_k\}_{\Delta'_r})^n$. Here $\Delta'_r = \{\delta'_1, \ldots, \delta'_r\}$ and $K\{\bar{x}\}_{\Delta'_r}$ denotes the Δ'_r-ring of Δ'_r-polynomials over K. Let

$$\bar{c} = \left(\bar{a}, D'\bar{a}, \ldots, D'^k\bar{a}, \frac{1}{g(\bar{a}, D'\bar{a} \ldots, D'^k\bar{a})}\right).$$

Let $X \subseteq F^{n(k+2)}$ be the Δ'_r-locus of $\bar{c}$ over K and $Y \subseteq F^{2n(k+2)}$ the Δ'_r-locus of $(\bar{c}, D'\bar{c})$ over K.

Claim. Y projects onto X.

Consider the Δ'_r-polynomial map $s(\bar{x}_0, \ldots, \bar{x}_{k+1}) : X \to F^{n(k+2)}$ given by

$$s = (\bar{x}_1, \ldots, \bar{x}_k, f\,\bar{x}_{k+1}, -\bar{x}^2_{k+1}\tau_{D'/\Delta'_r} g(\bar{x}_0, \ldots, \bar{x}_k, \bar{x}_1, \ldots, \bar{x}_k, f\,\bar{x}_{k+1}))$$

where any product between tuples is computed coordinatewise. Using (4.1), an easy computation shows $s(\bar{c}) = D'\bar{c}$. Given $\bar{b} \in X$, we note that $(\bar{b}, s(\bar{b})) \in Y$. Indeed, if h is a Δ'_r-polynomial over K vanishing at $(\bar{c}, D'\bar{c})$, then $h(\cdot, s(\cdot))$ vanishes at $\bar{c}$ and hence on all of X. So $(\bar{b}, s(\bar{b}))$ is in the Δ'_r-locus of $(\bar{c}, D'\bar{c})$ over K. That is, $(\bar{b}, s(\bar{b})) \in Y$. As this point projects onto $\bar{b}$ we have proven the claim.

Now, by induction, $(K, \Delta'_r) \models DCF_{0,r}$. Indeed, $(K, \Delta'_r) = \mathcal{K}_{r,N'}$ where N' is obtained from M by interchanging rows r and $(m+1)$. Hence, the claim implies that $Y(K)$ projects onto $X(K)$. Also, if $X(K) = \mathcal{V}(f_1, \ldots, f_s)$ where each f_i is a Δ'_r-polynomial, then clearly $Y(K) \subseteq \mathcal{V}(f_i, \tau_{D'/\Delta'_r} f_i : i = 1, \ldots, s)$. Hence, by condition (2), there is $\bar{d} \in X(K)$ such that $(\bar{d}, D'\bar{d}) \in Y(K)$.

Now, let $\rho(\bar{x})$ be the $\mathcal{L}_{r+1}$-formula over K obtained from ϕ by replacing each $\delta_1, \ldots, \delta_{r+1}$ for $d_{i,1}\delta_1 + \cdots + d_{i,r+1}\delta_{r+1}$, where $(d_{i,j}) \in \mathrm{SL}_{r+1}(\mathbb{Z})$ is the inverse matrix of M'. By construction, $\phi^{(K, \bar{\Delta}_r \cup \{\bar{D}\})} = \rho^{(K, \Delta'_r \cup \{D'\})}$. Thus it suffices to find a realisation of ρ in $(K, \Delta'_r \cup \{D'\})$. We may assume that the k of (4.1) is large enough so that we can write

$$\rho(\bar{x}) = \psi(\bar{x}, \delta_{r+1}\bar{x}, \ldots, \delta^k_{r+1}\bar{x})$$

where ψ is a conjunction of atomic $\mathcal{L}_r$-formulas over K. Let

$$\chi(\bar{x}_0, \ldots, \bar{x}_{k+1}, \bar{y}_0, \ldots, \bar{y}_{k+1}) := \psi(\bar{x}_0, \ldots, \bar{x}_k) \wedge \left(\wedge_{i=1}^k x_i = y_{i-1}\right).$$

Then $(F, \Delta'_r) \models \chi(\bar{c}, D'\bar{c})$, and so, as $(\bar{d}, D'\bar{d})$ is in the Δ'_r-locus of $(\bar{c}, D'\bar{c})$ over K, we have that $(F, \Delta'_r) \models \chi(\bar{d}, D'\bar{d})$. But since $\bar{d}$ is a K-point, we get $(K, \Delta'_r) \models \chi(\bar{d}, D'\bar{d})$. Writing the tuple $\bar{d}$ as $(\bar{d}_0, \ldots, \bar{d}_{r+1})$, we see that $\bar{d}_0$ is a realisation of ρ in $(K, \Delta'_r \cup \{D'\})$. This completes the proof. $\square$

Remark 4.4.

(1) Condition (2) of Theorem 4.3 is indeed first-order; expressible by an infinite collection of $\mathcal{L}_{m+1}$-sentences, one for each fixed choice of M, $f_1, \ldots, f_s$ and "shape" of W.

(2) In condition (2) we can strengthen the conclusion to ask for $\{\bar{a} \in V : (\bar{a}, D'\bar{a}) \in W\}$ to be Δ'-dense in V.

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Corrigendum to the paper:
"Geometric axioms for differentially closed fields with several commuting derivations"
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Omar León Sánchez
University of Waterloo

Abstract. We correct a small error found in the paper. While the mistake does not affect the main goal, it does render one of the theorems false as stated, and a correction is therefore called for.

In the proof of Lemma 2.6 (2) the iteration of the map τ was not performed properly and in fact the lemma is wrong; a counterexample is given by $f = \bar{x}_1$ and $k = 2$. This error does not, however, affect the geometric characterization given in Theorem 3.4 but only the attempt in Theorem 4.3 to express it as a first-order set of axioms. That attempt is incorrect; the main problem being that in general $\tau\mathcal{V}(f_1, \dots, f_s) \neq \mathcal{V}(f_1 \dots, f_s, \tau f_1, \dots, \tau f_s)$. But a different, indeed simpler, set of first-order axioms, which we will now describe, does express the geometric characterization.

Theorem 4.3′. Suppose $\Delta = \{\delta_1, \dots, \delta_m\}$ and $(K, \Delta \cup \{D\})$ is a characteristic zero differential field in $m+1$ commuting derivations. Then $(K, \Delta \cup \{D\}) \models DCF_{0,m+1}$ if and only if the following hold:

(1) $(K, \Delta) \models DCF_{0,m}$
(2) Suppose Λ is a characteristic set of a prime Δ-ideal of $K\{x_1, \dots, x_n\}$, O is a nonempty Δ-open subset of $\mathcal{V}(\Lambda)$ disjoint from $\mathcal{V}(H_\Lambda)$, and
$$W \subseteq \mathcal{V}(f, \tau f : f \in \Lambda)$$
is a Δ-closed set whose projection to $\mathcal{V}(\Lambda)$ contains O. Then there exists $\bar{a} \in O$ with $(\bar{a}, D\bar{a}) \in W$.

Remarks.

(i) Recall that H_Λ is the product of the separants and initials of the elements of Λ.
(ii) Condition (2) of 4.3′ is first-order expressible in the language of differential rings. Indeed, all that needs to be checked is that "$\Lambda = \{f_1, \dots, f_s\}$ is a characteristic set of a prime Δ-ideal of $K\{\bar{x}\}$" is a definable property on the coefficients of $f_1, \dots, f_s$. This is done by Tressl in §4 of [15] using Rosenfeld's criterion which reduces the problem to the classical problem of checking primality in polynomial rings in finitely many variables where uniform bounds are well-known.
(iii) These axioms for $DCF_{0,m+1}$ refer to $DCF_{0,m}$. Applying the theorem to the latter we have a similar characterization of $DCF_{0,m}$ in terms of $DCF_{0,m-1}$ plus a geometric axiom, and so on, until we get to $DCF_{0,0} := ACF_0$. That is, the theorem leads recursively to a full set of geometric axioms. Actually, it is possible to present these axioms all at once as one scheme by allowing linear combinations over the integers of the derivations (as was done in the

statement of the original Theorem 4.3, for example) but we have decided for the sake of clarity to present only the relative version in this corrigendum.

Proof of Theorem 4.3′. Suppose $(K, \Delta \cup \{D\})$ is differentially closed, and we are given Λ, $O \subseteq \mathcal{V}(\Lambda) \setminus \mathcal{V}(H_\Lambda)$, and $W \subseteq \mathcal{V}(f, \tau f : f \in \Lambda)$ satisfying the hypotheses of (2). By assumption Λ is a characteristic set of the prime Δ-ideal

$$[\Lambda] : H_\Lambda^\infty = \{f \in K\{\bar{x}\} : H_\Lambda^\ell f \in [\Lambda] \text{ for some } \ell\}.$$

Let $V := \mathcal{V}([\Lambda] : H_\Lambda^\infty)$, so V is an irreducible component of $\mathcal{V}(\Lambda)$ and $O \subseteq V$. Let $\widehat{W}$ be an irreducible component of W that projects Δ-dominantly onto V.

We claim that $\tau V|_O = \mathcal{V}(f, \tau f : f \in \Lambda)|_O$. Recall that, by definition, τV is $\mathcal{V}(f, \tau f : f \in \mathcal{I}(V/K))$. It is easy to see that $\mathcal{V}(f, \tau f : f \in \Lambda) = \mathcal{V}(f, \tau f : f \in [\Lambda])$. So, supposing that $(\bar{a}, \bar{b})$ is a root of f and τf for all $f \in [\Lambda]$, and $\bar{a} \in O$, we need to show that $(\bar{a}, \bar{b})$ is a root of τg for all $g \in \mathcal{I}(V/K)$. But $\mathcal{I}(V/K) = [\Lambda] : H_\Lambda^\infty$, so $H_\Lambda^\ell g \in [\Lambda]$ for some ℓ. We get

$$\begin{aligned} 0 &= \tau\big(H_\Lambda^\ell g\big)(\bar{a}, \bar{b}) \qquad \text{as } H_\Lambda^\ell g \in [\Lambda] \\ &= H_\Lambda^\ell(\bar{a})\tau g(\bar{a}, \bar{b}) + g(\bar{a})\tau(H_\Lambda^\ell)(\bar{a}, \bar{b}) \\ &= H_\Lambda^\ell(\bar{a})\tau g(\bar{a}, \bar{b}). \end{aligned}$$

Since O is disjoint from $\mathcal{V}(H_\Lambda)$ we have that $\tau g(\bar{a}, \bar{b}) = 0$, as desired.

It follows that a nonempty Δ-open subset of $\widehat{W}$ is contained in τV, and hence, by irreducibility, $\widehat{W} \subseteq \tau V$. We can now apply Theorem 3.4 (the geometric characterization of $DFC_{0,m+1}$) to $O \subseteq V$ and $\widehat{W} \subseteq \tau V$ to obtain $\bar{a} \in O$ such that $(\bar{a}, D\bar{a}) \in \widehat{W} \subseteq W$, as desired.

For the converse we suppose that (2) holds and we check the geometric characterization given in Theorem 3.4. That is, given irreducible Δ-closed sets $V \subseteq K^n$ and $W \subseteq \tau V$, with W projecting Δ-dominantly onto V, we need to find a point $\bar{a} \in V$ such that $(\bar{a}, D\bar{a}) \in W$.

Let Λ be a characteristic set of $\mathcal{I}(V/K)$ and let O be a nonempty Δ-open subset of $V \setminus \mathcal{V}(H_\Lambda) = \mathcal{V}(\Lambda) \setminus \mathcal{V}(H_\Lambda)$ that is contained in the projection of W (this is possible since W projects Δ-dominantly onto V and V is irreducible). Applying (2) to Λ, O and W, we obtain $\bar{a} \in O \subseteq V$ such that $(\bar{a}, D\bar{a}) \in W$. □

The precise changes required to make the paper formally correct are:

- Delete 2.6 (2), 2.7, 2.8 and 3.2 (which are false).
- In the proof of Remark 3.3 (1) drop the reference to 3.2 and use instead the fact that if V is defined over a Δ-subfield $F \leq K$ then $\mathcal{I}(V/K) = \mathcal{I}(V/F)K\{\bar{x}\}$.
- Replace Theorem 4.3 and its proof by the above Theorem 4.3′ and the proof given here.