A NOTE ON ISOLATED TYPES OF FINITE RANK

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ABSTRACT. Suppose T is totally transcendental and every minimal non-locally-modular type is nonorthogonal to a nonisolated minimal type over the empty set. It is shown that a finite rank type $p = \mathrm{tp}(a/A)$ is isolated if and only if $a \underset{Ab}{\perp} q(\mathcal{U})$ for every $b \in \mathrm{acl}(Aa)$ and $q \in S(Ab)$ nonisolated and minimal. This applies to the theory of differentially closed fields – where it is motivated by the differential Dixmier-Moeglin equivalence problem – and the theory of compact complex manifolds.

1. INTRODUCTION

Let T be a complete totally transcendental theory admitting elimination of imaginaries, and $\mathcal{U} \models T$ a sufficiently saturated model. We are interested in the following condition on a type $p \in S(A)$.

(†) Suppose $a \models p$, $b \in \mathrm{acl}(Aa)$, and $q \in S(Ab)$ is nonisolated and minimal. Then $a \underset{Ab}{\perp} q(\mathcal{U})$.

The condition is essentially about the relationship (or rather lack thereof) between p and the nonisolated minimal complete types of the theory – though the specific choice of parameters involved here are important. We will show that in certain theories of interest (including differentially closed fields and compact complex manifolds), and assuming that p is of finite rank, this condition characterises when p is isolated. It should be viewed as a reduction of the study of isolation from the finite rank case to the minimal case.

Another motivation for (†) comes from an application of the model theory of differentially closed fields of characteristic zero (DCF_0) to a problem in noncommutative algebra. It was observed in [2] that the classical Dixmier-Moeglin equivalence for noetherian algebras is connected to the relationship in DCF_0 between a finite rank type over constant parameters being isolated and being weakly orthogonal to the field of constants. The fact that these are not equivalent lead in [2] to the first counterexample to the Poisson Dixmier-Moeglin equivalence, and the first finite Gelfand-Kirillov dimension counterexample to the classical Dixmier-Moeglin equivalence. Now, $p \in S(A)$ being weakly orthogonal to the constants is precisely the instance of (†) when $b = \emptyset$ and q is the generic type of the constants. The counterexample in [2] was nonisolated and satisfied this instance of (†) but failed another instance; one where q was the generic type of a Manin kernel of a simple abelian variety not descending to the constants. So, the equivalence of (†) and isolation for

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finite rank types can be viewed as an abstract resolution to the Dixmier-Moeglin equivalence problem, where in order to get a true statement we have to replace weak orthogonality to the constants by all instances of (†).

Our main result here is the following: *Suppose that in T every minimal non-locally-modular type is nonorthogonal to a nonisolated minimal type over the empty set. Then a finite rank $p \in S(A)$ is isolated if and only if it satisfies* (†). This is Theorem 3.3 below. Specialising to the case of $T = \mathrm{DCF}_0$, we obtain in Theorem 4.4 a more concrete form which we point out yields a quick proof of the differential Dixmier-Moeglin equivalence for certain D-varieties that were considered in [3]. Also in §4 we give examples showing that our characterisation of isolation cannot be substantially improved, in that we really have to range over all $b \in \mathrm{acl}(Aa)$ in the formulation of (†).

We will use without extensive explanation various notions and facts from geometric stability theory – we suggest [12] as a general reference. The underlying total transcendentality assumption is so that prime models over sets exist and are unique up to isomorphism, and $p \in S(A)$ is isolated if and only if it is realised in a prime model over A. In particular, we freely use the following properties:

(1) $\mathrm{tp}(a/Ab)$ and $\mathrm{tp}(b/A)$ are isolated if and only if $\mathrm{tp}(ab/A)$ is isolated.
(2) $p = \mathrm{tp}(a/A)$ is isolated if and only if $\mathrm{stp}(a/A)$ is isolated.
(3) If p is nonisolated then so is any nonforking extension.

Points (1) and (2) follow easily using prime models. For point (3), note that if $q = \mathrm{tp}(a/B)$ is isolated by $\phi(x, b)$, and $r(y) := \mathrm{tp}(b/A)$ where $A = \mathrm{acl}(A) \subseteq B$ is such that q does not fork over A, then the $\phi(x, y)$-definition of $r(y)$ isolates $\mathrm{tp}(a/A)$. We also use the definable binding group theorem in totally transcendental theories.

2. Necessity of (†)

Without any assumptions beyond total transcendentality, we can show that (†) is necessary:

Proposition 2.1. *If $p \in S(A)$ is isolated then* (†) *holds.*

Proof. Note that $a \mathop{\smile\!\!\!\!\!\!\,|}_C q(\mathcal{U})$ where $C = \mathrm{dcl}(Aba) \cap \mathrm{dcl}(Abq(\mathcal{U}))$. Indeed, this follows from stable embeddedness, see for example the Appendix of [5]. So it suffices to show that $C \subseteq \mathrm{acl}(Ab)$.

Let M be a prime model over Ab. Since $\mathrm{tp}(a/A)$ and $\mathrm{tp}(b/Aa)$ are isolated, so is $\mathrm{tp}(a/Ab)$. By automorphisms, we may assume that a is in M. It follows that $C \subseteq \mathrm{dcl}(Aba) \subseteq M$.

Now, let $c \in C$. Since $c \in \mathrm{dcl}(Abq(\mathcal{U}))$ we can write $c = f(e)$ where f is Ab-definable and e is a finite tuple from $q(\mathcal{U})$. Let $\phi \in q$ witness the Morley rank and degree of q, and let $X := \phi(\mathcal{U})$. (Note that a minimal type need not have Morley rank 1.) Since $c \in M$ and X is M-definable, there must exist e' from $X(M)$ such that $c = f(e')$ as well. We claim that $e' \mathop{\smile\!\!\!\!\!\!\,|}_{Ab} e$. This will suffice, as then $\mathrm{dcl}(Abe) \cap \mathrm{dcl}(Abe') \subseteq \mathrm{acl}(Ab)$, and hence $c = f(e) = f(e')$ is in $\mathrm{acl}(Ab)$, as desired.

To prove that $e' \mathop{\smile\!\!\!\!\!\!\,|}_{Ab} e$, note that by minimality of q we have $e \in \mathrm{acl}(Abe_1 \ldots e_r)$ where $(e_1, \ldots, e_r)$ is a subtuple of e forming an acl-basis for e over Ab, and we need only show that $e' \mathop{\smile\!\!\!\!\!\!\,|}_{Ab} (e_1, \ldots, e_r)$. If not, then, writing $e' = (e'_1, \ldots, e'_n)$, for some

$i = 0, \ldots, n-1$, we must have $(e'_1, \ldots, e'_i) \underset{Ab}{\perp} (e_1, \ldots, e_r)$ but $e'_{i+1} \underset{Abe'_1 \ldots e'_i}{\not\perp} (e_1, \ldots, e_r)$. So by minimality of q there is $j \leq r$ such that

$$e_j \in \mathrm{acl}(Abe'_1 \ldots e'_i e'_{i+1} e_1 \ldots e_{j-1}).$$

On the other hand,

$$e_j \underset{Ab}{\perp} (e'_1, \ldots, e'_i, e_1, \ldots, e_{j-1})$$

since $(e_1, \ldots, e_j) \underset{Ab}{\perp} (e'_1, \ldots, e'_i)$ and $e_j \underset{Ab}{\perp} (e_1, \ldots, e_{j-1})$. Using the above two displayed facts we can compute that

$$\begin{aligned} \mathrm{RM}(q) &= \mathrm{RM}(e_j / Abe'_1 \ldots e'_i e_1 \ldots e_{j-1}) \\ &\leq \mathrm{RM}(e'_{i+1} / Abe'_1 \ldots e'_i e_1 \ldots e_{j-1}) \\ &\leq \mathrm{RM}(e'_{i+1} / Ab). \end{aligned}$$

As e'_{i+1} is in X which witnesses the Morley rank and degree of q, we must have $\mathrm{tp}(e'_{i+1}/Ab) = q$. But $e'_{i+1} \in M$, and so we have a realisation of q in the prime model over Ab, contradicting the fact that q is nonisolated. □

Remark 2.2. The proof of the above proposition actually proves something stronger. If $p = \mathrm{tp}(a/A)$ is isolated then (†) holds with the following two strengthenings:

(a) We can range over all b such that $\mathrm{tp}(b/Aa)$ is isolated, rather than asking for $b \in \mathrm{acl}(Aa)$.

(b) We can allow $q \in S(Ab)$ to be any (nonisolated) strongly regular type, rather than asking for it to be minimal. See [4, §6.4] for strong regularity.

But what we are really interested in is not strong consequences of isolation, but rather weak sufficient conditions.

Question 2.3. Does (†) characterise isolation of a *finite rank* type $p \in S(A)$?

3. Sufficiency of (†)

We will make an additional assumption on T, satisfied by the theory of differentially closed fields and the theory of compact complex manifolds, under which (†) becomes also a sufficient condition for isolation of finite rank types. But before doing so, let us record the following known fact whose proof was pointed out to us by Anand Pillay (see also [12, Lemma 7.2.7]).

Lemma 3.1. *Suppose $p = \mathrm{tp}(a/A)$ is nonorthogonal to a locally modular minimal type. Then there is $e \in \mathrm{acl}(Aa)$ with $U(e/A) = 1$.*

Proof. Let q be a locally modular minimal type such that p is nonorthogonal to q, and let Q be the set of A-conjugates of q. So Q is A-invariant and p is not foreign to Q. It follows that there exists $c \in \mathrm{dcl}(Aa) \setminus \mathrm{acl}(A)$ such that $\mathrm{tp}(c/A)$ is Q-internal – see, for example, [12, Corollary 7.4.6]. Local modularity of q, and hence of each type in Q, implies that $\mathrm{tp}(c/A)$ is 1-based – see, for example, [13, Corollary 9]. Now, let $B \supseteq A$ be such that $U(c/B) = U(c/A) - 1$, and let $e = \mathrm{Cb}(c/B)$. By 1-basedness, $e \in \mathrm{acl}(Ac) \subseteq \mathrm{acl}(Aa)$. Also, as $c \underset{Ae}{\perp} B$,

$$U(c/B) + 1 = U(c/A) = U(ce/A) = U(c/Ae) + U(e/A) = U(c/B) + U(e/A),$$

so that $U(e/A) = 1$. □

Our additional assumption on T is as follows.

Assumption 3.2. Every complete non-locally-modular minimal type is nonorthogonal to a nonisolated minimal type in $S(\emptyset)$.

This is satisfied in DCF_0 and CCM because of the particular manifestations of the Zilber dichotomy in these theories: In CCM every non-locally-modular minimal type is nonorthogonal to the generic type of the projective line, and in DCF_0 every non-locally-modular minimal type is nonorthogonal to the generic type of the field of constants. These types are nonisolated and minimal as the projective line and the field of constants, respectively, are 0-definable strongly minimal sets with infinitely many points in $\operatorname{acl}(\emptyset)$.

Theorem 3.3. *Suppose T is totally transcendental and satisfies Assumption 3.2. Let $p \in S(A)$ be of finite rank. Then p is isolated if and only if it satisfies (†).*

Proof. That isolated types satisfy (†) in arbitrary totally transcendental theories is the content of Proposition 2.1.

Before proving the converse, we first observe that if $\operatorname{tp}(a/A)$ satisfies (†) then for any $e \in \operatorname{acl}(Aa)$ so do $\operatorname{tp}(e/A)$ and $\operatorname{tp}(a/Ae)$. For the former, let $b \in \operatorname{acl}(Ae)$, and $q \in S(Ab)$ nonisolated and minimal. Then as $b \in \operatorname{acl}(Aa)$ also, (†) implies that $a \underset{Ab}{\perp} q(\mathcal{U})$, and hence $e \underset{Ab}{\perp} q(\mathcal{U})$. To see that $\operatorname{tp}(a/Ae)$ satisfies (†), let $b \in \operatorname{acl}(Aea)$ and $q \in S(Aeb)$ nonisolated and minimal. Then as $eb \in \operatorname{acl}(Aa)$ and $\operatorname{tp}(a/A)$ satisfies (†), we get $a \underset{Aeb}{\perp} q(\mathcal{U})$, as desired.

We proceed to prove by induction on $U(p)$ that if p satisifies (†) then it is isolated. If $U(p) = 0$ then it is isolated. If $U(p) = 1$ then isolation is immediate from (†) applied with $b = \emptyset$. So assume that $U(p) > 1$.

Suppose $p = \operatorname{tp}(a/A)$ is orthogonal to all non-locally-modular minimal types. Since p is of finite rank it is nonorthogonal to some minimal type, which by assumption must be locally modular. So by Lemma 3.1, there exists $e \in \operatorname{acl}(Aa)$ with $U(e/A) = 1$. Since $U(p) > 1$, $a \notin \operatorname{acl}(Ae)$. Hence both $U(e/A)$ and $U(a/Ae)$ are less than $U(p)$, and as we have observed, both $\operatorname{tp}(e/A)$ and $\operatorname{tp}(a/Ae)$ satisfy (†). By induction, they are both isolated. Hence $\operatorname{tp}(a/A)$ is isolated, as desired.

It remains to consider the case when p is nonorthogonal to some non-locally-modular minimal type. By Assumption 3.2, p is nonorthogonal to a nonisolated minimal type in $S(\emptyset)$. Taking the nonforking extension of that type to A we get $q \in S(A)$ minimal, nonisolated, and nonorthogonal to p. Applying (†) with $b = \emptyset$, we have that $a \perp_A q(\mathcal{U})$. On the other hand, nonorthogonality implies that there exists $d \in \operatorname{dcl}(Aa) \setminus A$ with $\operatorname{tp}(d/A)$ internal to q, see [12, Corollary 7.4.6]. We know that both $\operatorname{tp}(d/A)$ and $\operatorname{tp}(a/Ad)$ satisfy (†). If $U(d/A) < U(p)$ then, by induction, both $\operatorname{tp}(d/A)$ and $\operatorname{tp}(a/Ad)$ are isolated; and consequently p would be isolated. So we may assume $U(d/A) = U(p)$. That is, a and d are interalgebraic over A. Let $p' = \operatorname{stp}(d/A)$ and G be the q-binding group of p'. This is a definable group over $\operatorname{acl}(A)$ acting definably over A on $p'(\mathcal{U})$. Moreover, $a \perp_A q(\mathcal{U})$ implies $d \perp_A q(\mathcal{U})$, so that the binding group acts transitively on $p'(\mathcal{U})$. But this implies that $p'(\mathcal{U})$ is definable, and so by automorphisms is $\operatorname{acl}(A)$-definable. That is, p' is isolated. But $a \in \operatorname{acl}(Ad)$ now implies that $\operatorname{stp}(a/A)$, and hence $\operatorname{tp}(a/A) = p$, is isolated. □

4. The case of DCF_0

We begin this section with a pair of examples in DCF_0 that show Theorem 3.3 to be best possible in the sense that it is essential in (†) to consider arbitrary $b \in \mathrm{acl}(Aa)$. That is, as Example 4.1 shows, one cannot deduce isolation of $p = \mathrm{tp}(a/A)$ by checking that $a \underset{A}{\perp} q(\mathcal{U})$ for all nonisolated minimal types $q \in S(A)$. In fact, as Example 4.2 shows, it does not even suffice to consider $q \in S(Ab)$ for all $b \in \mathrm{dcl}(Aa)$, one must pass to $\mathrm{acl}(Aa)$.

Example 4.1 (Parametrised family of Manin kernels)**.** From [2, §4] one sees that there exist nonisolated finite rank types $p = \mathrm{tp}(a)$ in DCF_0 satisfying:

- p is weakly orthogonal to the field of constants $\mathcal{C}$; and,
- there exists $b \in \mathrm{dcl}(a)$ such that $\mathrm{tp}(b)$ is minimal $\mathcal{C}$-internal and $\mathrm{tp}(a/b)$ is minimal nontrivial locally modular.

We claim that $a \perp q(\mathcal{U})$ for any $q \in S(\emptyset)$ nonisolated and minimal.

Proof. Indeed, if q is trivial then it is orthogonal to both $\mathrm{tp}(b/A)$ and $\mathrm{tp}(a/b)$, and hence to p, so that $a \perp q(\mathcal{U})$ follows. If q is non-locally-modular then it is the generic type of some strongly minimal definable set X over A (see [12, §2.3]). By nonisolation of q we must have that $X \cap \mathrm{acl}(\emptyset)$ is infinite. But then $X(\mathcal{C})$ is infinite, and hence cofinite. So $q(\mathcal{U}) \subseteq \mathcal{C}^n$ and $a \perp q(\mathcal{U})$ follows by weak orthogonality. As there are no minimal nontrivial locally modular types over the empty set in DCF_0, these are all the possibilities for q. (It is not important here that we worked over the empty set; such examples exist over any $A \subseteq \mathcal{C}$.) □

Example 4.2 (The symmetric power of the j-function equation)**.** Freitag and Scanlon [6] have studied the order three algebraic differential equation satisfied by the analytic j-function. It defines, over the empty set, a strongly minimal trivial set X in DCF_0. But unlike all previous such examples, X is not ω-categorical. Indeed, $X \cap \mathrm{acl}(c)$ is infinite for any $c \in X$ generic. This is due to Hecke correspondences, see the final paragraph of the proof of [6, Theorem 4.7]. Now, let $c_1, c_2 \in X$ be a pair of independent generics. Let a be a code for $\{c_1, c_2\}$, and set $p := \mathrm{tp}(a)$. We claim that p is nonisolated but $a \underset{b}{\perp} q(\mathcal{U})$ for any $b \in \mathrm{dcl}(a)$ and any nonisolated minimal $q \in S(b)$.

Proof. Note, first of all, that $\mathrm{tp}(c_1/c_2)$ is not isolated since it is the generic type of X over c_2, and $X \cap \mathrm{acl}(c_2)$ is infinite by construction. In particular, as a is interalgebraic with (c_1, c_2), we have that p is nonisolated.

Now, because p is the code of a set of independent generic elements in a trivial strongly minimal set, we have by [11, Example 2.2] that p admits no *proper fibrations*. That is, if $b \in \mathrm{dcl}(a)$ then either $a \in \mathrm{acl}(b)$ or $b \in \mathrm{acl}(\emptyset)$. So it suffices to show that $a \perp q(\mathcal{U})$ for any nonisolated minimal type q over $\mathrm{acl}(\emptyset)$. Suppose this fails for some q. Then $(c_1, c_2) \not\perp q(\mathcal{U})$. Since q is minimal, this implies that q is nonorthogonal to the generic type of X. In particular, $\mathrm{RM}(q) = 1$. (Indeed, after taking a nonforking extension, a realisation of q is interalgebraic with a generic element of the strongly minimal set X.) It follows that q is the generic type of some strongly minimal definable set Y over $\mathrm{acl}(\emptyset)$. Moreover, Y must be trivial since X is. But then, $Y(\mathcal{C})$ is finite, so that $Y \cap \mathrm{acl}(\emptyset)$ is finite, contradicting the fact that q is nonisolated. □

Theorem 3.3 can be viewed as giving an understanding of finite rank isolated types in terms of minimal isolated types. In practice, we often know a lot about the nontrivial minimal types. For example, in differentially closed fields they are nonorthogonal to the constants or to Manin kernels. (We suggest [8] for an exposition on the Manin kernels associated to abelian varieties.) The nonisolated ones are even more closely linked to these two cases.

Lemma 4.3. *Supose* $T = \mathrm{DCF}_0$. *A nontrivial minimal type* $p \in S(A)$ *is nonisolated if and only if* $p(\mathcal{U}) \subseteq \mathrm{acl}(Aq(\mathcal{U}))$ *where* $q \in S\big(\mathrm{acl}(A)\big)$ *is the generic type of either the constant field or of the Manin kernel of some simple abelian variety over* $\mathrm{acl}(A)$ *that does not descend to the constants.*

Proof. First of all, as we have said, p being a nontrivial minimal type means that p is nonorthogonal to the generic type q of either either the constant field or of the Manin kernel of some simple abelian variety that does not descend to the constants. In the former case, as $\mathcal{C}$ is defined over the empty set, we can take q to be the generic type of $\mathcal{C}$ over $\mathrm{acl}(A)$. It is maybe less well known that in the latter case too we can take the simple abelian variety to be over $\mathrm{acl}(A)$, and hence $q \in S\big(\mathrm{acl}(A)\big)$. This can, however, be deduced from the results in [7].[1] We give a few details. By [7, Lemma 2.11], p is nonorthogonal to the Manin kernel of some simple abelian variety G_1 over Aa_1 for some a_1. Now take a conjugate a_2 of a_1 that is independent of a_1 over A. We then get a Manin kernel of some simple abelian variety G_2 over Ab_2 such that p is also nonorthogonal to its generic type. Hence, the two Manin kernels are nonorthogonal. By [7, Theorem 2.12], G_1 and G_2 are isogenous. By the Claim in the proof of [7, Proposition 2.8], it follows that G_1 is isogenous to a (necessarily simple) abelian variety G over $\mathrm{acl}(A)$. The Manin kernel of G is thus nonorthogonal to that of G_1 (by Theorem 2.12 of [7] again). So p is nonorthogonal to the Manin kernel of G, which is over $\mathrm{acl}(A)$, as desired.

Now, fixing such $q \in S\big(\mathrm{acl}(A)\big)$, suppose $p(\mathcal{U}) \not\subseteq \mathrm{acl}(Aq(\mathcal{U}))$. By minimality of p, this means that there is $a \models p$ such that $a \mathop{\smile\!\!\!\!\!\;|}_{A} q(\mathcal{U})$. On the other hand, p is nonorthogonal to q, which by minimality of p means that p is almost internal to q. These conditions, namely $a \mathop{\smile\!\!\!\!\!\;|}_{A} q(\mathcal{U})$ and $\mathrm{tp}(a/A)$ almost internality to q, imply that p is isolated (see the argument in the last paragraph of the proof of Theorem 3.3).

For the converse, note that the generic type of the constant field $\mathcal{C}$ is minimal and nonisolated since $\mathcal{C}$ is a strongly minimal set with infinitely many points in $\mathrm{acl}(A)$, that infinite set being the characteristic zero field $\mathcal{C} \cap \mathrm{acl}(A)$. The generic type of the Manin kernel of a simple abelian variety over $\mathrm{acl}(A)$ is minimal and nonisolated for the same reason – the Manin kernel is strongly minimal and it has infinitely many $\mathrm{acl}(A)$-points, namely the torsion points of the abelian variety. This gives the right-to-left direction, using for example (†) applied with $b = \emptyset$, which holds of any isolated type p by Proposition 2.1. □

We obtain the following improvement of Theorem 3.3 in the case of DCF_0.

Theorem 4.4. *Suppose* $T = \mathrm{DCF}_0$ *and* $p = \mathrm{tp}(a/A)$ *is of finite rank. Then* p *is isolated if and only if the following hold:*

(i) $a \mathop{\smile\!\!\!\!\!\;|}_{A} \mathcal{C}$*; and*

[1] We are grateful to Martin Hils for pointing this out to us.

(ii) *for every* $b \in \operatorname{acl}(Aa)$ *and* G *a simple abelian variety over* $\operatorname{acl}(Ab)$ *that does not descend to the constants, letting* $G^\sharp$ *denote the Manin kernel of* G, $a \mathop{\downarrow}_{Ab} G^\sharp$*; and*

(iii) *for every* $b \in \operatorname{acl}(Aa)$ *and* $q \in S(Ab)$ *nonisolated and trivial minimal,* $a \mathop{\downarrow}_{Ab} q(\mathcal{U})$*.*

Proof. Suppose p is isolated. By Proposition 2.1 we know that (†) holds. So, for every $b \in \operatorname{acl}(Aa)$ and $q \in S(Ab)$ nonisolated and minimal, $a \mathop{\downarrow}_{Ab} q(\mathcal{U})$. Taking q to be nonisolated minimal trivial yields (iii). Taking q to be the generic type of $G^\sharp$ over $\operatorname{acl}(Ab)$ we have that $q(\mathcal{U}) = G^\sharp \setminus \operatorname{acl}(Ab)$ and hence $a \mathop{\downarrow}_{Ab} G^\sharp$. Similarly, taking $b = \emptyset$ and q to be the generic type of $\mathcal{C}$ over A, we have that $q(\mathcal{U}) = \mathcal{C} \setminus \operatorname{acl}(A)$ and hence $a \mathop{\downarrow}_{A} \mathcal{C}$.

For the converse, by Theorem 3.3, it suffices to show that (i) through (iii) imply (†). Let $b \in \operatorname{acl}(Aa)$ and $q \in S(Ab)$ nonisolated and minimal. We want to show $a \mathop{\downarrow}_{Ab} q(\mathcal{U})$. If q is trivial then this follows by (iii). If q is nontrivial then by Lemma 4.3 we have that $q(\mathcal{U}) \subseteq \operatorname{acl}(Abq'(\mathcal{U}))$ where $q' \in S(Ab)$ is the generic type of either the constant field or of the Manin kernel of a simple abelian variety over $\operatorname{acl}(Ab)$ that does not descend to the constants. So it suffices to show that $a \mathop{\downarrow}_{Ab} q'(\mathcal{U})$. If q' is the generic type of a Manin kernel then this is (ii). So suppose q' is the generic type of $\mathcal{C}$ over Ab. For any finite tuple c from $q'(\mathcal{U}) \subseteq \mathcal{C}^n$ we have $a \mathop{\downarrow}_{A} c$ by (i), and so $a \mathop{\downarrow}_{Ab} c$ as $b \in \operatorname{acl}(Aa)$. We have shown that $a \mathop{\downarrow}_{Ab} q'(\mathcal{U})$, as desired. □

In practice, conditions (i) and (ii) of Theorem 4.4 are relatively easy to check as they refer to concrete differential varieties. It is the trivial case, namely condition (iii), that remains in general intractable. Unfortunately, we cannot eliminate this condition, even when $A = \emptyset$. For example[2], let X be the trivial strongly minimal but not ω-categorical 0-definable set coming from [6] and discussed in Example 4.2 above. Let (c_1, c_2) be an independent pair of generic elements of X, and this time let $a := (c_1, c_2)$ and $p := \operatorname{tp}(a)$. Then p satisfies conditions (i) and (ii) of Theorem 4.4, but fails fails condition (iii) with $b = c_2$. Indeed, $q := \operatorname{tp}(c_1/c_2)$ is minimal and trivial as it is the generic type of X, it is nonisolated since X has infinitely many points algebraic over c_2, and clearly $a \mathop{\not\downarrow}_{Ac_2} c_1$.

Nonetheless, in some cases we can ignore condition (iii); for example when p extends a finite rank definable group, since in that case any trivial minimal type is orthogonal to p so that (iii) is automatic. In other cases we can eliminate (ii) as well; for example, if p is analysable in the constants then it will be orthogonal to all Manin kernels as well as all trivial minimal types.

[2]We have to resort here to this relatively recently discovered example because all previously known trivial strongly minimal sets in DCF_0 were ω-categorical, and it is easy to see that the generic type q of an ω-categorical strongly minimal set over a differential field that is finitely generated over its constant subfield is always isolated, and hence cannot pose an obstacle to (iii).

4.1. **Connection to the Dixmier-Moeglin equivalence.** In [2, §9] it was shown that finite rank types $p = \mathrm{tp}(a/A)$ in DCF_0, with $A \subseteq \mathcal{C}$, that satisfy condition (i) of Theorem 4.4 but are not isolated, can be used to produce finite Gelfand-Kirillov dimension counterexamples to the classical Dixmier-Moeglin equivalence for noetherian algebras. The types with these properties that arose there were the ones coming from the parameterised Manin kernels of Example 4.1. As we have just seen, the generic type of $X \times X$, where X is defined by the differential equation satisfied by the j-function, gives us another such counterexample, different from the ones appearing in [2].

The *differential* Dixmier-Moeglin equivalence for D-varieties was made explicit in [3], where positive results, as opposed to counterexamples, were the focus. The following is Corollary 2.4 of [3]. It was used there to show that D-groups over the constants satisfy the differential Dixmier-Moeglin equivalence, and eventually to verify the classical Dixmier-Moeglin equivalence for Hopf Ore extensions. We give here an alternative proof; indeed it is an immediate consequence of Theorem 4.4 above. We use freely the terminology of D-varieties, and the specific notions developed in [3], without further explanation.

Corollary 4.5. *Suppose (V, s) is a D-variety over an algebraically closed δ-subfield A of the field of constants, with the property that every irreducible D-subvariety of V over A is compound isotrivial. Then (V, s) satisfies the differential Dixmier-Moeglin equivalence.*

Proof. Working over the constants the Dixmier-Moeglin equivalence reduces to showing that every type in $S(A)$ extending $(V, s)^\sharp$, that is weakly orthogonal to the constants, is isolated. The compound isotriviality assumption means that every such type $p = \mathrm{tp}(a/A)$ is analysable in the constants. Conditions (ii) and (iii) of Theorem 4.4 are therefore automatically satisfied. Hence, by Theorem 4.4, weak orthogonality to $\mathcal{C}$, which is condition (i), implies isolation. □

5. The case of CCM

As we have mentioned, Theorem 3.3 also applies to the theory of compact complex manifolds. Much of what was done in the previous section for DCF_0 goes through for CCM if one replaces $\mathcal{C}$ by the projective line and Manin kernels by *nonstandard simple complex tori* (see [1]) of dimension greater than 1. This is especially the case if you restrict attention to compact Kähler manifolds where one has essential saturation (see [9]). There is even an analogue of the algebraic differential equation satisfied by the analytic j-function: in [10] it is observed that there exists a 0-definable strongly minimal trivial set in CCM that is not ω-categorical. However, there is one key obstacle to obtaining a full analogue of Theorem 4.4. We do not know if the analogue of the Claim in the proof of [7, Proposition 2.8] holds:

Question 5.1. Suppose G_1 and G_2 are nonstandard simple complex tori over b_1 and b_2 respectivey, and b_1 and b_2 are independent over $c \in \mathrm{dcl}(b_1) \cap \mathrm{dcl}(b_2)$. Is it the case that if G_1 and G_2 are isogenous then there is $c' \in \mathrm{acl}(c)$ and a nonstandard simple complex torus over c' to which both G_1 and G_2 are isogenous?

In the algebraic case one uses the fact that an algebraically closed set is a model, which is no longer true in CCM. Without a positive answer to this question one obtains only a weak analogue of Theorem 4.4 where in condition (ii) Manin kernels are replaced by nonisolated minimal types that are nonorthogonal to the generic type of a nonstandard simple complex tori defined possibly over extra parameters.

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