A CORRIGENDUM TO “$D$-GROUPS AND THE DME”

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The purpose of this note, which is not intended for publication, is to fill a small gap that appears in the proof of Proposition 3.5 of [1]. At the bottom of page 364 of the published version there is a mistake: when addressing the case of $n = 1$; that is, the case when $a$ and $\log \delta a := \frac{\delta a}{a}$ are algebraically dependent over $k$; it is claimed that something forces the polynomial $P_0(x)$ to be trivial. In fact, $P_0(x) = d(1 - x)$ for any $d \in k$ is possible, and should have been dealt with.

Instead of following the details in [1] we will give a direct and more conceptual proof of Proposition 3.5 of [1] in this special case:

**Proposition.** Suppose $k$ is a field of characteristic zero, $R$ is a commutative affine Hopf $k$-algebra, and $\delta$ is a $k$-linear derivation on $R$ that is an $a$-coderivation for some group-like $a \in R$. If $a$ and $\log \delta a$ are algebraically dependent over $k$ then $\log \delta a = d(1 - a)$ for some $d \in k$.

In particular, the conclusion of [1, Proposition 3.5] holds with $c := -\frac{\log \delta a}{n}$.

**Proof.** There is an affine algebraic group $G$, a nontrivial character $a : G \to \mathbb{G}_m$, and an $a$-twisted $D$-group structure $s : G \to TG$, all over $k$, such that $R = k[G]$ and $\delta$ is the derivation on $k[G]$ induced by $s$.

Consider the map $\pi := (a, \log \delta a) : G \to \mathbb{G}_m \ltimes \mathbb{G}_a$ that appears in [1] where it is shown, by an easy computation at the beginning of the proof of Proposition 3.8, that, since $a$ is a character and $s$ is $a$-twisted, $\pi$ is a morphism of algebraic groups. Since $\{a, \log \delta a\}$ is algebraically dependent over $k$, and $a \neq 1$, the (connected) algebraic subgroup $H := \pi(G) \leq \mathbb{G}_m \ltimes \mathbb{G}_a$ must be 1-dimensional. Hence the coordinate projection $\pi_1 : H \to \mathbb{G}_m$ is surjective with finite kernel. It follows that $\pi_1$ factors as $\rho_n \phi$, for some $n > 0$, where $\phi : H \to \mathbb{G}_m$ is an isomorphism of algebraic groups and $\rho_n : \mathbb{G}_m \to \mathbb{G}_m$ is given by $\rho_n(x) = x^n$. Let $\mathbb{G}_m \ltimes_n \mathbb{G}_a$ be the semidirect product where $\mathbb{G}_m$ acts on $\mathbb{G}_a$ by $(x, y) \mapsto x^n y$.

We first claim that $F : H \to \mathbb{G}_m \ltimes_n \mathbb{G}_a$ given by $(x, y) \mapsto (\phi(x, y), y)$, is a group homomorphism. Indeed,

$$F((x, y)(x', y')) = F(xx', y + xy')$$
$$= (\phi(xx', y + xy'), y + xy')$$
$$= (\phi(x, y)\phi(x', y'), y + xy')$$
$$= \phi(x, y)\phi(x', y') + \phi(x, y) + \phi(x, y)y'$$
$$= \phi(x, y)\phi(x', y') + \phi(x, y)y'$$
$$= \phi(x, y)\phi(x', y')$$
$$= \phi(x, y) F(x', y')$$

as desired.

Next, we claim that $\chi := \pi_2 \phi^{-1} : \mathbb{G}_m \to \mathbb{G}_a$ is a $\rho_n$-twisted additive character. That is, that $\chi(xx') = \chi(x) + x^n \chi(x')$. Indeed, note that the image $H' = F(H)$ is
the graph of $\chi$. So for $x, x' \in \mathbb{G}_m$ we have $(xx', \chi(xx')) \in H'$. But as $F$ is a group homomorphism, $H'$ is a subgroup of $\mathbb{G}_m \ltimes_{n} \mathbb{G}_a$, and so 

$$(x, \chi(x))(x', \chi(x')) = (xx', \chi(x) + x^n \chi(x')) \in H',$$

as well. It follows that $\chi(xx') = \chi(x) + x^n \chi(x')$, as desired.

Choose $b \in k$ such that $b^n \neq 1$. We have, for $x \in \mathbb{G}_m$, 

$$\chi(xb) = \chi(x) + x^n \chi(b) \quad \text{and} \quad \chi(bx) = \chi(b) + b^n \chi(x)$$

so that $\chi(x)(1 - b^n) = \chi(b)(1 - x^n)$. Letting $d := \frac{\chi(b)}{1 - b^n}$ we get $\chi(x) = d(1 - x^n)$.

So, for all $g \in G$, we have 

$$\log \delta a(g) = \pi_2 \pi(g) = \chi \phi \pi(g) \quad \text{as} \quad \chi = \pi_2 \phi^{-1} = d(1 - \phi \pi(g)^n) = d(1 - \pi_1 \pi(g)) \quad \text{as} \quad \pi_1 = \rho_n \phi = d(1 - a(g)).$$

That is, $\log \delta a = d(1 - a)$, as desired.

For the “in particular” clause, a direct computation shows that for $c := \frac{-d^2}{2}$ we have the identity $a^2\delta a = \frac{3}{2}(\delta a)^2 + c(a^2 - a^4)$ as claimed in Proposition 3.5 of [1].

It may be worth pointing out that what $\log \delta a = d(1 - a)$ says geometrically is that we have the short exact sequence 

$$1 \longrightarrow (\ker(a), u) \longrightarrow (G, s) \overset{a}{\longrightarrow} (\mathbb{G}_m, t_d) \longrightarrow 1$$

where $u := s \mid_{\ker(a)}$ makes $(\ker(a), u)$ a $D$-group, $t_d(x) := d(x - x^2)$ makes $(\mathbb{G}_m, t_d)$ an id-twisted $D$-group, and the morphisms are algebraic group homomorphisms that are also morphisms of $D$-varieties.

References