Quick course in Universal Algebra and Tame Congruence Theory

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Workshop on Universal Algebra and the Constraint Satisfaction Problem

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(with revisions added after the presentation)
Outline

0. Apology

Part I: Basic universal algebra

1. Algebras, term operations, varieties
2. Congruences
3. Classifying algebras by congruence properties
4. The abelian/nonabelian dichotomy

Part II: Tame congruence theory

5. Polynomial subreducts
6. Minimal sets and traces (of a minimal congruence)
7. The 5-fold classification and types
8. Classifying algebras by the types their varieties omit
1. Algebras, term operations, varieties

An algebra: $A = (A; F) = (\text{universe}; \{\text{fundamental operations}\})$

term: any formal expression built from [names for] the fundamental operations and variables

terms in $n$ variables define $n$-ary term operations of $A$.

**Definition**

The clone of $A$ is $\text{Clo}(A) = \{\text{all term operations of } A\} = \langle F \rangle$.

$\text{Clo}(A)$ is the fundamental invariant of $A$.

**Definition**

$A, B$ are term-equivalent if they have the same universe and same term functions.
Definition

- $f : A^n \to A$ is **idempotent** if $f(x, x, \ldots, x) = x \quad \forall x \in A$.
- $A = (A, F)$ is **idempotent** if every $f \in F$ (equivalently, $f \in \langle F \rangle$) is idempotent.

CSP'ers care only about idempotent algebras.

This tutorial is **not** specifically focussed on idempotent algebras.

Oh well.
Varieties

Definition
A class of algebras is

- **equational** if it can be axiomatized by **identities**, i.e. (universally quantified) equations between terms.
- a **variety** if it is closed under forming homomorphic images (H), subalgebras (S), and products (P).

Basic theorems

1. (G. Birkhoff) Varieties = equational classes.
2. (Tarski) The smallest variety \( \text{var}(\mathcal{K}) \) containing \( \mathcal{K} \) is \( \text{var}(\mathcal{K}) = \text{HSP}(\mathcal{K}) \).

\( \text{var}(A) \), the **variety generated by** \( A \), is another useful invariant of \( A \).
2. Congruences

Suppose $A, B$ are algebras “in the same language” and $\sigma : A \to B$ is a homomorphism.

The pre-images of $\sigma$ partition $A$.

**Definition**

- $\ker(\sigma) =$ the equivalence relation on $A$ given by this partition.
- **congruence** of $A$: any kernel of a homomorphism with domain $A$.

Alternatively: congruences of $A$ are the equivalence relations $\theta$ on $A$ which
  - Are compatible with $F$ ($\forall f \in F, a \sim^\theta a' \Rightarrow f(a, b, \ldots) \sim^\theta f(a', b, \ldots)$, etc.)
  - Support a natural construction of $A/\theta$ on the $\theta$-classes.
Definition

Con(\(A\)) = \{set of all congruences of \(A\}\).

\((\text{Con}(\(A\)), \subseteq)\) is a poset with top = \(A^2\) and bottom = \{(a, a) : a \in A\} . . .

[Picture]

. . . and is a lattice: any two \(\theta, \varphi\) have a g.l.b. (meet) and a l.u.b. (join):

\[
\begin{align*}
\theta \land \varphi &= \theta \cap \varphi \\
\theta \lor \varphi &= \text{transitive closure of } \theta \cup \varphi \\
&= \{\text{all } (a, b) \text{ connected by alternating } \theta, \varphi\text{-paths}\}.
\end{align*}
\]

\((\text{Con}(\(A\)); \land, \lor)\) is a surprisingly useful invariant of \(A\).
3. Classifying algebras by congruence properties

Distributive law (for lattices): \( x \land (y \lor z) = (x \land y) \lor (x \land z) \) and dually.

Modular law: distributive law restricted to non-antichain triples \((x, y, z)\).

**Definition**

<table>
<thead>
<tr>
<th>Say ( A ) is</th>
<th>if ( \text{Con}(A) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>congruence distributive (CD)</td>
<td>is distributive</td>
</tr>
<tr>
<td>congruence modular (CM)</td>
<td>is modular</td>
</tr>
<tr>
<td>congruence permutable (CP)</td>
<td>satisfies ( x \circ y = y \circ x )</td>
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</tbody>
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First approx. to \( \theta \lor \varphi \):

\[
\theta \circ \varphi \overset{\text{def}}{=} \{(a, c) : \exists b, a \sim b \sim c\}.
\]

**Fact**: For an algebra \( A \), TFAE and imply CM:

- \( \theta \lor \varphi = \theta \circ \varphi \quad \forall \theta, \varphi \in \text{Con}(A) \).
- \( \theta \circ \varphi = \varphi \circ \theta \quad \forall \theta, \varphi \in \text{Con}(A) \).
**A connection:** existence of term operations satisfying certain identities ⇔ congruence lattice properties. For example:

**Definition**

Let $m(x, y, z)$ be a 3-ary term for $A$.

- $m$ is a **majority** (or **3-NU**) term for $A$ if
  
  $$A \models m(x, x, y) \approx m(x, y, x) \approx m(y, x, x) \approx x.$$  

- $m$ is a **Mal’tsev** term for $A$ if
  
  $$A \models m(x, x, y) \approx m(y, x, x) \approx y.$$  

**Examples**

- Using lattice ops, $m(x, y, z) := (x \lor y) \land (x \lor z) \land (y \lor z)$ is 3-NU.
- Using group ops, $m(x, y, z) := x \cdot y^{-1} \cdot z$ (or $x - y + z$) is Mal’tsev.
Theorem

- A has a 3-NU term $\Rightarrow$ every $B \in \text{var}(A)$ is CD.
- A has a Mal’tsev term $\Leftrightarrow$ every $B \in \text{var}(A)$ is CP.

Proof of 2nd item (Mal’tsev term $\Leftrightarrow$ $\text{var}(A)$ is CP).

($\Rightarrow$). Let $m(x, y, z)$ be a Mal’tsev term for $A$. Let $B \in \text{var}(A)$ and $\theta, \varphi \in \text{Con}(B)$. It suffices to show $\theta \circ \varphi \subseteq \varphi \circ \theta$. Assume $(a, c) \in \theta \circ \varphi$, say $a \thicksim b \not\sim c$.

$m$ is also a Mal’tsev term for $B$, so

$$a = m(a, c, c) \not\sim m(a, b, c) \thicksim m(a, a, c) = c$$

witnessing $(a, c) \in \varphi \circ \theta$.

Key: $m(x, y, z)$ gives a uniform witness to $\theta \circ \varphi \subseteq \varphi \circ \theta$. 
We construct a generic instance of $\theta \circ \varphi \subseteq \varphi \circ \theta$ in $\var{A}$.

Let $B = F_{\var{A}}(x, y, z) \in \var{A}$, the free $\var{A}$-algebra of rank 3

$\theta = \text{the smallest congruence of } B \text{ containing } (x, y)$
$\varphi = \text{the smallest congruence of } B \text{ containing } (y, z)$.

Clearly $x \theta \sim y \varphi \sim z$, so $(x, z) \in \theta \circ \varphi$.

Assuming $\var{A}$ is CP, then $(x, z) \in \varphi \circ \theta$.

Choose a witness $m \in B$, so $x \varphi \sim m \theta \sim z$.

$m \text{ "is" a term.}$

$(x, m) \in \varphi \text{ implies } \var{A} \models x \approx m(x, z, z)$

$(m, z) \in \theta \text{ implies } \var{A} \models m(x, x, z) \approx z.$
Commentary on 1st item (3-NU term $\Rightarrow$ every $\var(A)$ is CD).

Theorem (B. Jónsson)

Given $A$, TFAE:

- $\var(A)$ is CD.
- Every $\text{Con}(B) \models \alpha \cap (\beta \circ \gamma) \subseteq (\alpha \cap \beta) \lor (\alpha \cap \gamma)$
- $\exists k$ such that every $\text{Con}(B) \models$

$$\alpha \cap (\beta \circ \gamma) \subseteq (\alpha \cap \beta) \circ (\alpha \cap \gamma) \circ (\alpha \cap \beta) \circ \cdots \circ (\alpha \cap [\beta \mid \gamma])$$

Call the displayed condition $\text{CD}(k)$.

Exercise

$\var(A) \models \text{CD}(2) \iff A$ has a 3-NU term.

Remark: $\text{CD}(3)$ is witnessed by a pair of 3-ary terms, etc. (Called Jónsson terms)
4. The abelian/nonabelian dichotomy

Definition

An algebra $A$ is abelian if the diagonal $0_A := \{(a, a) : a \in A\}$ is a block of some congruence of $A^2$.

Equivalently, if for all term operations $f(\bar{x}, \bar{y})$,

$$\forall \bar{a}, \bar{b}, \bar{c}, \bar{d} : f(\bar{a}, \bar{c}) = f(\bar{a}, \bar{d}) \rightarrow f(\bar{b}, \bar{c}) = f(\bar{b}, \bar{d}).$$

Examples: abelian groups; $R$-modules; $G$-sets.

Non-examples: nonabelian groups; anything with a semilattice operation.

By restricting the quantifiers in $(*)$, can define notion of a congruence being abelian; or of one congruence centralizing another. Leads to notions of solvability, nilpotency.
Nicest setting: in CM varieties.

- Abelian algebras (and congruences) are affine (see below).

**Definition**

A is **affine** if (i) \( A \) has a Mal’tsev term \( m(x, y, z) \), and (ii) all fundamental operations commute with \( m(x, y, z) \).

Equivalently, if there is a ring \( R \), an \( R \)-module \( R \mathcal{M} \) with universe \( A \), and a submodule \( U \leq_{\mathcal{A}} R \times_{\mathcal{A}} R \mathcal{M} \) such that

\[
\text{Clo } A = \text{ all } \sum_{i=1}^{n} r_{i}x_{i} + a \quad (r_{i} \in R, \ a \in A)
\]

for which \((1 - \sum_{i=1}^{n} r_{i}, a) \in U.\)

(In which case \( m(x, y, z) = x - y + z. \))

(Idempotent case: \( U = \{(0, 0)\}. \))
Still in CM varieties:

- Centralizer relation on congruences is understood.
- Abelian-free intervals in $\text{Con}(A)$ correspond to structure “similar to” that in CD varieties.
- Thus we get positive information on either side of the abelian/nonabelian dichotomy.
- Example (Freese, McKenzie, 1981): Let $A$ be a finite algebra in a CM variety. Whether or not $\text{var}(A)$ is residually finite can be characterized by centralizer facts in $\text{HS}(A)$. 
PART II: Tame Congruence Theory

5. Polynomial subreducts
6. Minimal sets and traces (of a minimal congruence)
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8. Classifying algebras by the types their varieties omit
5. Polynomial subreducts

**Polynomial operations**: like term operations, but allowing parameters.

**Definition**

Algebras $A$, $B$ are **polynomially equivalent** if they have the same universe and the same *polynomial* operations.

- Polynomial equivalence is coarser than term-equivalence.
- Example: on the set $2 := \{0, 1\}$, there are exactly 7 algebras up to polynomial equivalence.

[picture on next slide]
(2,+) (2,\wedge,\vee,\neg) (2,\wedge,\vee) (2,\neg) (2,\emptyset)

2 3 4 5

1
Strange construction #1.

**Definition**

Let $A = (A, \mathcal{F})$ be an algebra and $S \subseteq A$. Form a new algebra with

- universe $= S$
- clone of operations $= \text{all } f|_{S^n}, f \text{ an } n\text{-ary polynomial operation of } A$ with $f(S^n) \subseteq S$.

This is $A|_S$, the **polynomial algebra induced on** $S$ (by $A$).
Toy example

\[ A = \text{1-dimensional vector space over finite field } F = GF(p^n) \]
\[ = (F; \{+, (\lambda x)_{\lambda \in F}\}) \]

Polynomial operations of \( A \): all

\[ \sum_{i=1}^{n} \lambda_i x_i + a, \quad \lambda_i \in F, \ a \in F \]

Let \( S = F^* = F \setminus \{0\} \). Then

Exercise

- Every nonconstant operation of \( A|_S \) depends on exactly one variable.
- \( A|_S \) is term-equivalent to a \( G \)-set with all constants, where \( G \) is the cyclic group of order \( p^n - 1 \).
Another toy example

\[ A = (A_4, \cdot), \text{ the alternating group on 4 letters.} \]

Recall: \( |A_4| = 12 \), and the elements include the 8 permutations of \( \{1, 2, 3, 4\} \) which cycle 3 elements, the 3 permutations which match each element with a partner and switch partners, and the identity permutation.

Polynomial operations of \( A \): rather more complicated.

Let \( N = \) its 4-element normal subgroup.

**Group Theory Exercise**

\( A|_N \) is term-equivalent to a 1-dimensional vector space over \( GF(4) \) with all constants.

In both examples, the point is that \( A|_S \) is not a subalgebra of \( A \), or even of the same type of algebra as \( A \).
Strange construction #2. Let $A$ be a finite algebra.

**Definition**

- $E(A) = \{\text{all unary polynomials } e(x) \text{ of } A \text{ satisfying } e(e(x)) = e(x)\}$.
- **Neighborhood**: any $e(A), e \in E(A)$.

Let $\alpha \in \text{Con}(A)$ be a **minimal** (nonzero) congruence.

**Definition**

- $\mathcal{N}_A(\alpha) = \{\text{those neighborhoods which intersect at least one } \alpha\text{-block in } \geq 2 \text{ points}\}$.
- **$\alpha$-minimal set**: any **minimal** member of $\mathcal{N}_A(\alpha)$ (with respect to $\subseteq$).
- **$\alpha$-trace**: any intersection of an $\alpha$-minimal set with an $\alpha$-block, provided the intersection has $\geq 2$ points.
- **$\alpha$-body**: the union of all $\alpha$-traces in one $\alpha$-minimal set.
Example: the group $A_4$. [Picture]

Let $N$ be the 4-element normal subgroup and $\alpha = \theta_N$ the congruence whose classes are the three cosets of $N$.

Fix an element $a$ of order 3. (So the cosets of $N$ are $N, aN, a^2N$.)

Consider the following unary polynomials of $A_4$.

\[
e_1(x) = x^4, \quad U_1 = e_1(A_4)
\]
\[
e_2(x) = a(a(ax^4)^4)^4, \quad U_2 = e_2(A_4)
\]
\[
e_3(x) = a(a^{-1}x)^3, \quad U_3 = e_3(A_4).
\]

$e_1(x) = x$ for all $x \in aN \cup a^2N$ while $e_1(x) = 1$ for $x \in N$. Hence $e_1(e_1(x)) = e_1(x)$ and $U_1 = \{1\} \cup aN \cup a^2N$ is a neighborhood.

$e_2$ maps $N$ to 1, $aN$ to $a$ and $a^2N$ to $a^2$. Hence $e_2(e_2(x)) = e_2(x)$ and $U_2 = \{1, a, a^2\}$ is a neighborhood. (In fact, every transversal of the cosets of $N$ is a neighborhood.)
Example (continued)

\[ e_3(x) = x \text{ for } x \in aN \text{ while } e_3(x) = a \text{ for } x \in N \cup a^2N. \] Hence
\[ e_3(e_3(x)) = e_3(x) \text{ and } U_3 = aN \text{ is a neighborhood.} \]

\[ U_2 \notin \mathcal{N}_A(\alpha) \text{ since } U_2 \text{ does not meet any } \alpha\text{-class nontrivially. } U_1 \in \mathcal{N}_A(\alpha) \text{ but } U_1 \text{ is not an } \alpha\text{-minimal set because } U_3 \in \mathcal{N}_A(\alpha) \text{ and } U_3 \subset U_1. \] A
computer can show that \( U_3 \) is an \( \alpha\)-minimal set. In fact, the cosets of \( N \) are precisely the \( \alpha\)-minimal sets.

Since each \( \alpha\)-minimal set in this example is contained entirely inside an \( \alpha\)-class, the \( \alpha\)-traces and \( \alpha\)-bodies are identical to the \( \alpha\)-minimal sets, i.e., the cosets of \( N \).

**Warning:** this is not the typical picture!!

[Typical picture on next page]
This portrays the classes of a minimal congruence $\alpha$ (dashed lines), one $\alpha$-minimal set (dark black line), and an $\alpha$-body consisting of two $\alpha$-traces (red lines). The two points not in the body comprise the tail.
7. The 5-fold classification and types

**Key step:** focus on polynomial algebras induced on $\alpha$-minimal sets and $\alpha$-traces.

The latter are catalogued up to polynomial equivalence.

**Theorem 1 (P. P. Pálfy)**

Let $A$ be a finite algebra, $\alpha$ a minimal congruence, and $N$ an $\alpha$-trace. Then the induced polynomial algebra $A|_N$ is polynomially equivalent to one of:

1. a $G$-set.
2. a 1-dimensional vector space over a finite field.
3. a 2-element boolean algebra.
4. a 2-element lattice $(L, \{\wedge, \vee\})$.
5. a 2-element semilattice $(S, \lor)$.
Theorem 2

Let $A$ be a finite algebra and $\alpha$ a minimal congruence. If $N_1, N_2$ are any two $\alpha$-traces, then $A|_{N_1} \cong A|_{N_2}$.

Thus we get a 5-fold classification of minimal congruences.

Definition

For $\alpha$ a minimal congruence of $A$, the type of $\alpha$ is the common type

- Type 1 (unary)
- Type 2 (vector space)
- Type 3 (boolean)
- Type 4 (lattice)
- Type 5 (semilattice)

of the polynomial algebras induced on the $\alpha$-traces of $A$.

Example: The minimal congruence of the group $A_4$ has “Type 2.”
The 5 types reflect 5 distinct “local” structures in an algebra $A$.

In turn, the local structure reflects and is reflected by the global structure of $A$.

The lowest-order tool is:

**Theorem 3**

Let $\alpha$ be a minimal congruence of $A$.

(Connectedness) Each nontrivial $\alpha$-block is the union of connected $\alpha$-traces.

Moreover, $A$ has enough unary polynomials to:

(Isomorphism) . . . map any $\alpha$-trace isomorphically to any other.

(Density) . . . map any two distinct elements in an $\alpha$-block to distinct elements of an $\alpha$-trace.
Connecting local and global structure

Example:

**Theorem**

Let \( \alpha \) be a minimal congruence of finite \( A \).

- \( \alpha \) is abelian \( \iff \) the type of \( \alpha \) is 1 or 2.

- If \( \alpha \) is abelian and \( A \) is idempotent, then each block of \( \alpha \) (as a subalgebra of \( A \)) is **quasi-affine** (i.e., is a subalgebra of a reduct of a module).
Typing $\text{Con}(A)$

Suppose $\alpha \in \text{Con}(A)$ is not minimal. Choose $\delta < \alpha$ so that $\alpha$ is minimal over $\delta$.

[Picture]

Passing to $A/\delta$, we can assign a type (1–5) to the pair $(\delta, \alpha)$.

In this way, a type (1–5) is assigned to each edge of the graph of $\text{Con}(A)$. Much is known.

In applications, one often needs local information about $(\delta, \alpha)$ in $A$ (not just in $A/\delta$).

Leads to a refined def. of $(\delta, \alpha)$-minimal sets, traces and bodies ($\subseteq A$).

Polynomial algebras induced on $(\delta, \alpha)$-traces are completely understood in types 3 and 4, largely understood in cases 1, 2 and 5.
Classifying algebras by the types their varieties omit

**Definition**

Let $A$ be a finite algebra and $i \in \{1, 2, 3, 4, 5\}$. We say that $\text{var}(A)$

- **admits** type $i$ if type $i$ occurs in $\text{Con}(B)$ for some (finite) $B \in \text{var}(A)$. (WLOG, $B \leq A^n$.)
- **omits** type $i$ otherwise.

Omitting types gives us another way to classify $\text{var}(A)$. For example:

**Theorem**

For $A$ finite, TFAE:

- $A$ has a Taylor term (equivalently, a weak NU-term).
- $\text{var}(A)$ omits type 1.

Similar characterizations (via terms satisfying equations) exist for $\text{var}(A)$ omitting any “down-set” of types (e.g., $\{1, 2\}$, $\{1, 5\}$, etc).
Fitting in the congruence classifications

**Theorem**

- \( \text{var}(A) \) is CM iff \( \text{var}(A) \) omits type 1 and 5 and has no tails.
- \( \text{var}(A) \) is CD iff \( \text{var}(A) \) omits type 1, 2 and 5 and has no tails.

Another relevant class is the class of \( A \) for which \( \text{var}(A) \) omits the abelian types 1,2. This class is characterized as those \( A \) for which every \( \text{Con}(B) \) satisfies a certain implicational law called SD(\( \land \)).

[Big picture on next page]
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Tame Congruence Theory  
Nashville, June 2007  

"modules"  
CM  
CP  
Omits type 1  
Omits types 1,5  
CD  
NU  
Omit types 1,2  
"semilattices"  
"lattices"  
"boolean algebras"  
"groups"  
"rings"  
"modules"  
"G-sets"  
"groups"  
"rings"  
"modules"  
Omits types 1,5  
Omits type 1  
"semilattices"  
"lattices"  
"boolean algebras"
Postscript (not included in lecture):

Tame congruence theory reveals how far is the gap between those idempotent $A$ for which $\text{CSP}(A)$ is known to be NP-complete ($\text{var}(A)$ admits type 1), and those for which $\text{CSP}(A)$ is known to be in P (CP, NU, bounded width, varieties generated by a finite conservative algebra).

Tame congruence theory suggests intermediate classes of algebras to be explore.

Under weak assumptions (e.g., that $\text{var}(A)$ omits type 1), the theory yields subtle, positive structural information about all $B \in \text{var}(A)$.

Most importantly, the theory suggests one way to localize, divide and conquer the confusion of “all finite algebras.”