Quick course in Universal Algebra and Tame Congruence Theory

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Workshop on Universal Algebra and the Constraint Satisfaction Problem

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(with revisions added after the presentation)

Outline

0. Apology

PART I: Basic universal algebra

1. Algebras, term operations, varieties
2. Congruences
3. Classifying algebras by congruence properties
4. The abelian/nonabelian dichotomy

PART II: Tame congruence theory

5. Polynomial subreducts
6. Minimal sets and traces (of a minimal congruence)
7. The 5-fold classification and types
8. Classifying algebras by the types their varieties omit

1. Algebras, term operations, varieties

An algebra: $\mathbf{A} = (A;\, F)$
$= (\textit{universe};\, \{\textit{fundamental operations}\})$

term: any formal expression built from [names for] the fundamental operations and variables

terms in n variables define n-ary **term operations** of $\mathbf{A}$.

Definition

The **clone of A** is $\mathrm{Clo}(\mathbf{A}) = \{\text{all term operations of } \mathbf{A}\} = \langle F \rangle$.

$\mathrm{Clo}(\mathbf{A})$ is the fundamental invariant of $\mathbf{A}$.

Definition

$\mathbf{A}, \mathbf{B}$ are **term-equivalent** if they have the same universe and same term functions.

Definition

- $f : A^n \to A$ is **idempotent** if $f(x, x, \dots, x) = x \quad \forall x \in A$.
- $\mathbf{A} = (A, F)$ is **idempotent** if every $f \in F$ (equivalently, $f \in \langle F \rangle$) is idempotent.

CSP'ers care only about idempotent algebras.

This tutorial is **not** specifically focussed on idempotent algebras.

Oh well.

Varieties

Definition

A class of algebras is

- **equational** if it can be axiomatized by **identities**, i.e. (universally quantified) equations between terms.
- a **variety** if it is closed under forming homomorphic images (**H**), subalgebras (**S**), and products (**P**).

Basic theorems

1. (G. Birkhoff) Varieties = equational classes.
2. (Tarski) The smallest variety $\mathrm{var}(\mathcal{K})$ containing $\mathcal{K}$ is $\mathrm{var}(\mathcal{K}) = \mathbf{HSP}(\mathcal{K})$.

$\mathrm{var}(\mathbf{A})$, the *variety generated by* **A**, is another useful invariant of **A**.

2. Congruences

Suppose $\mathbf{A}, \mathbf{B}$ are algebras "in the same language" and $\sigma : \mathbf{A} \to \mathbf{B}$ is a homomorphism.

[Picture]

The pre-images of σ partition A.

Definition

- $\ker(\sigma)$ = the equivalence relation on A given by this partition.
- **congruence** of $\mathbf{A}$: any kernel of a homomorphism with domain $\mathbf{A}$.

Alternatively: congruences of $\mathbf{A}$ are the equivalence relations θ on A which

- Are compatible with F ($\forall f \in F$, $a \stackrel{\theta}{\sim} a' \Rightarrow f(a, b, \ldots) \stackrel{\theta}{\sim} f(a', b, \ldots)$, etc.)
- Support a natural construction of $\mathbf{A}/\theta$ on the θ-classes.

Definition

$\mathrm{Con}(\mathbf{A}) = \{$set of all congruences of $\mathbf{A}\}$.

$(\mathrm{Con}(\mathbf{A}), \subseteq)$ is a poset with top $= A^2$ and bottom $= \{(a,a) : a \in A\}$ ...

[Picture]

...and is a **lattice**: any two θ, φ have a g.l.b. (**meet**) and a l.u.b. (**join**):

$$\begin{aligned} \theta \wedge \varphi &= \theta \cap \varphi \\ \theta \vee \varphi &= \text{transitive closure of } \theta \cup \varphi \\ &= \{\text{all } (a,b) \text{ connected by alternating } \theta, \varphi\text{-paths}\}. \end{aligned}$$

$(\mathrm{Con}(\mathbf{A}); \wedge, \vee)$ is a surprisingly useful invariant of $\mathbf{A}$.

3. Classifying algebras by congruence properties

Distributive law (for lattices): $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and dually.

Modular law: distributive law restricted to non-antichain triples (x, y, z).

Definitiion

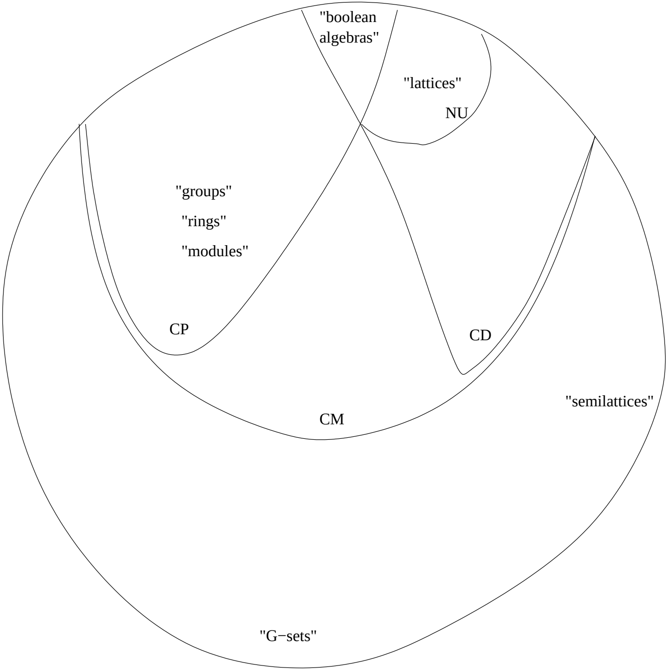
Say **A** is		if $\mathrm{Con}(\mathbf{A})$
congruence distributive	(CD)	is distributive
congruence modular	(CM)	is modular
congruence permutable	(CP)	satisfies $x \circ y = y \circ x$

First approx. to $\theta \vee \varphi$:

$$\theta \circ \varphi \stackrel{def}{=} \{(a, c) : \exists b,\ a \stackrel{\theta}{\sim} b \stackrel{\varphi}{\sim} c\}.$$

Fact: For an algebra **A**, TFAE and imply CM:

- $\theta \vee \varphi = \theta \circ \varphi \quad \forall \theta, \varphi \in \mathrm{Con}(\mathbf{A})$.
- $\theta \circ \varphi = \varphi \circ \theta \quad \forall \theta, \varphi \in \mathrm{Con}(\mathbf{A})$.


"boolean
algebras"
"lattices"
NU
"groups"
"rings"
"modules"
CP
CD
CM
"semilattices"
"G−sets"

A connection: existence of term operations satisfying certain identities $\Leftrightarrow$ congruence lattice properties. For example:

Definition

Let $m(x, y, z)$ be a 3-ary term for **A**.

- m is a **majority** (or **3-NU**) term for **A** if

$$\mathbf{A} \models m(x, x, y) \approx m(x, y, x) \approx m(y, x, x) \approx x.$$

- m is a **Mal'tsev** term for **A** if

$$\mathbf{A} \models m(x, x, y) \approx m(y, x, x) \approx y.$$

Examples

- Using lattice ops, $m(x, y, z) := (x \vee y) \wedge (x \vee z) \wedge (y \vee z)$ is 3-NU.
- Using group ops, $m(x, y, z) := x \cdot y^{-1} \cdot z$ (or $x - y + z$) is Mal'tsev.

Theorem

- $\mathbf{A}$ has a 3-NU term $\Rightarrow$ every $\mathbf{B} \in \mathrm{var}(\mathbf{A})$ is CD.
- $\mathbf{A}$ has a Mal'tsev term $\Leftrightarrow$ every $\mathbf{B} \in \mathrm{var}(\mathbf{A})$ is CP.

Proof of 2nd item (Mal'tsev term $\Leftrightarrow$ $\mathrm{var}(\mathbf{A})$ is CP).

($\Rightarrow$). Let $m(x, y, z)$ be a Mal'tsev term for $\mathbf{A}$. Let $\mathbf{B} \in \mathrm{var}(\mathbf{A})$ and $\theta, \varphi \in \mathrm{Con}(\mathbf{B})$. It suffices to show $\theta \circ \varphi \subseteq \varphi \circ \theta$. Assume $(a, c) \in \theta \circ \varphi$, say $a \overset{\theta}{\sim} b \overset{\varphi}{\sim} c$.

m is also a Mal'tsev term for $\mathbf{B}$, so

$$a = m(a, c, c) \overset{\varphi}{\sim} m(a, b, c) \overset{\theta}{\sim} m(a, a, c) = c$$

witnessing $(a, c) \in \varphi \circ \theta$.

Key: $m(x, y, z)$ gives a **uniform witness** to $\theta \circ \varphi \subseteq \varphi \circ \theta$.

($\Leftarrow$). We construct a generic instance of $\theta \circ \varphi \overset{?}{\subseteq} \varphi \circ \theta$ in $\mathrm{var}(\mathbf{A})$.

Let $\mathbf{B} = \mathbb{F}_{\mathrm{var}(\mathbf{A})}(x, y, z) \in \mathrm{var}(\mathbf{A})$, the free $\mathrm{var}(\mathbf{A})$-algebra of rank 3
$\theta =$ the smallest congruence of $\mathbf{B}$ containing (x, y)
$\varphi =$ the smallest congruence of $\mathbf{B}$ containing (y, z).

Clearly $x \overset{\theta}{\sim} y \overset{\varphi}{\sim} z$, so $(x, z) \in \theta \circ \varphi$.

Assuming $\mathrm{var}(\mathbf{A})$ is CP, then $(x, z) \in \varphi \circ \theta$.

Choose a witness $m \in B$, so $x \overset{\varphi}{\sim} m \overset{\theta}{\sim} z$.

m "is" a term.

$(x, m) \in \varphi$ implies $\mathrm{var}(\mathbf{A}) \models x \approx m(x, z, z)$

$(m, z) \in \theta$ implies $\mathrm{var}(\mathbf{A}) \models m(x, x, z) \approx z$.

Commentary on 1st item (3-NU term $\Rightarrow$ every $\mathrm{var}(\mathbf{A})$ is CD).

Theorem (B. Jónsson)

Given **A**, TFAE:

- $\mathrm{var}(\mathbf{A})$ is CD.
- Every $\mathrm{Con}(\mathbf{B}) \models \alpha \cap (\beta \circ \gamma) \subseteq (\alpha \cap \beta) \vee (\alpha \cap \gamma)$
- $\exists k$ such that every $\mathrm{Con}(\mathbf{B}) \models$

$$\alpha \cap (\beta \circ \gamma) \subseteq \underbrace{(\alpha \cap \beta) \circ (\alpha \cap \gamma) \circ (\alpha \cap \beta) \circ \cdots \circ (\alpha \cap [\beta|\gamma])}_{k}$$

Call the displayed condition $\mathrm{CD}(k)$.

Exercise

$\mathrm{var}(\mathbf{A}) \models \mathrm{CD}(2) \Leftrightarrow$ **A** has a 3-NU term.

Remark: $\mathrm{CD}(3)$ is witnessed by a pair of 3-ary terms, etc. (Called *Jónsson* terms)

4. The abelian/nonabelian dichotomy

Definition

An algebra $\mathbf{A}$ is **abelian** if the diagonal $0_A := \{(a, a) : a \in A\}$ is a block of some congruence of $\mathbf{A}^2$.

Equivalently, if for all term operations $f(\bar{x}, \bar{y})$,

$$\forall \bar{a}, \bar{b}, \bar{c}, \bar{d} : f(\bar{a}, \bar{c}) = f(\bar{a}, \bar{d}) \rightarrow f(\bar{b}, \bar{c}) = f(\bar{b}, \bar{d}). \qquad (*)$$

Examples: abelian groups; R-modules; G-sets.

Non-examples: nonabelian groups; anything with a semilattice operation.

By restricting the quantifiers in $(*)$, can define notion of a *congruence* being abelian; or of one congruence **centralizing** another. Leads to notions of solvability, nilpotency.

Nicest setting: in CM varieties.

- Abelian algebras (and congruences) are *affine* (see below).

Definition

A is **affine** if (i) **A** has a Mal'tsev term $m(x, y, z)$, and (ii) all fundamental operations commute with $m(x, y, z)$.

Equivalently, if there is a ring R, an R-module ${}_RM$ with universe A, and a submodule $U \leq {}_RR \times {}_RM$ such that

$$\text{Clo}\,\mathbf{A} = \text{all} \sum_{i=1}^{n} r_i x_i + a \quad (r_i \in R,\ a \in A)$$

$$\text{for which } \left(1 - \sum_{i=1}^{n} r_i,\ a\right) \in U.$$

(In which case $m(x, y, z) = x - y + z$.)
(Idempotent case: $U = \{(0, 0)\}$.)

Still in CM varieties:

- Centralizer relation on congruences is understood.
- Abelian-free intervals in $\mathrm{Con}(\mathbf{A})$ correspond to structure "similar to" that in CD varieties.
- Thus we get positive information on either side of the abelian/nonabelian dichotomy.
- Example (Freese, McKenzie, 1981): Let $\mathbf{A}$ be a finite algebra in a CM variety. Whether or not $\mathrm{var}(\mathbf{A})$ is residually finite can be characterized by centralizer facts in $\mathbf{HS}(\mathbf{A})$.

PART II: Tame Congruence Theory

5. Polynomial subreducts
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5. Polynomial subreducts

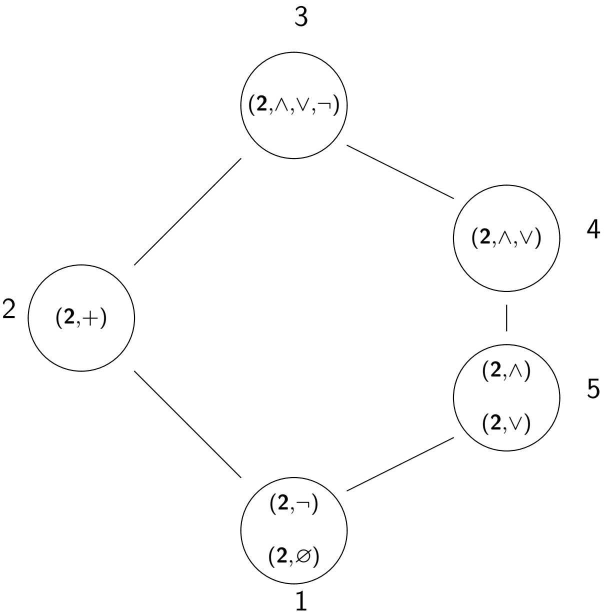
Polynomial operations: like term operations, but allowing parameters.

Definition

Algebras **A**, **B** are **polynomially equivalent** if they have the same universe and the same *polynomial* operations.

- Polynomial equivalence is coarser than term-equivalence.
- Example: on the set $\mathbf{2} := \{0, 1\}$, there are exactly 7 algebras up to polynomial equivalence.

  [picture on next slide]


3
$(\mathbf{2}, \wedge, \vee, \neg)$
4
$(\mathbf{2}, \wedge, \vee)$
2
$(\mathbf{2}, +)$
$(\mathbf{2}, \wedge)$
5
$(\mathbf{2}, \vee)$
$(\mathbf{2}, \neg)$
$(\mathbf{2}, \varnothing)$
1

Strange construction #1.

Definition

Let $\mathbf{A} = (A, \mathcal{F})$ be an algebra and $S \subseteq A$. Form a new algebra with

- universe $= S$
- clone of operations $=$ all $f|_{S^n}$, f an n-ary **polynomial** operation of $\mathbf{A}$ with $f(S^n) \subseteq S$.

This is $\mathbf{A}|_S$, the **polynomial algebra induced on** S (by $\mathbf{A}$).

Toy example

$$\begin{aligned} \mathbf{A} &= \text{1-dimensional vector space over finite field } F = GF(p^n) \\ &= (F;\ \{+,\ (\lambda x)_{\lambda \in F}\}) \end{aligned}$$

Polynomial operations of **A**: all

$$\sum_{i=1}^{n} \lambda_i x_i + a, \quad \lambda_i \in F,\ a \in F$$

Let $S = F^* = F \setminus \{0\}$. Then

Exercise

- Every nonconstant operation of $\mathbf{A}|_S$ depends on exactly one variable.
- $\mathbf{A}|_S$ is term-equivalent to a G-set with all constants, where G is the cyclic group of order $p^n - 1$.

Another toy example

$$\mathbf{A} = (A_4, \cdot), \text{ the alternating group on 4 letters.}$$

Recall: $|A_4| = 12$, and the elements include the 8 permutations of $\{1, 2, 3, 4\}$ which cycle 3 elements, the 3 permutations which match each element with a partner and switch partners, and the identity permutation.

Polynomial operations of $\mathbf{A}$: rather more complicated.

Let $N =$ its 4-element normal subgroup.

Group Theory Exercise

$\mathbf{A}|_N$ is term-equivalent to a 1-dimensional vector space over $GF(4)$ with all constants.

In both examples, the point is that $\mathbf{A}|_S$ is **not** a subalgebra of $\mathbf{A}$, or even of the same type of algebra as $\mathbf{A}$.

6. Minimal sets and traces (of a minimal congruence)

Strange construction #2. Let **A** be a finite algebra.

Definition

- $E(\mathbf{A}) = \{$all unary polynomials $e(x)$ of **A** satisfying $e(e(x)) = e(x)\}$.
- **Neighborhood**: any $e(A)$, $e \in E(\mathbf{A})$.

Let $\alpha \in \mathrm{Con}(\mathbf{A})$ be a **minimal** (nonzero) congruence.

Definition

- $\mathcal{N}_{\mathbf{A}}(\alpha) = \{$those neighborhoods which intersect at least one α-block in ≥ 2 points$\}$.
- α-**minimal set**: any *minimal* member of $\mathcal{N}_{\mathbf{A}}(\alpha)$ (with respect to $\subseteq$).
- α-**trace**: any intersection of an α-minimal set with an α-block, provided the intersection has ≥ 2 points.
- α-**body**: the union of all α-traces in one α-minimal set.

Example: the group A_4. [Picture]

Let N be the 4-element normal subgroup and $\alpha = \theta_N$ the congruence whose classes are the three cosets of N.

Fix an element a of order 3. (So the cosets of N are N, aN, a^2N.)

Consider the following unary polynomials of A_4.

$$e_1(x) = x^4 \qquad U_1 = e_1(A_4)$$
$$e_2(x) = a(a(ax^4)^4)^4 \qquad U_2 = e_2(A_4)$$
$$e_3(x) = a(a^{-1}x)^3 \qquad U_3 = e_3(A_4).$$

$e_1(x) = x$ for all $x \in aN \cup a^2N$ while $e_1(x) = 1$ for $x \in N$. Hence $e_1(e_1(x)) = e_1(x)$ and $U_1 = \{1\} \cup aN \cup a^2N$ is a neighborhood.

e_2 maps N to 1, aN to a and a^2N to a^2. Hence $e_2(e_2(x)) = e_2(x)$ and $U_2 = \{1, a, a^2\}$ is a neighborhood. (In fact, every transversal of the cosets of N is a neighborhood.)

Example (continued)

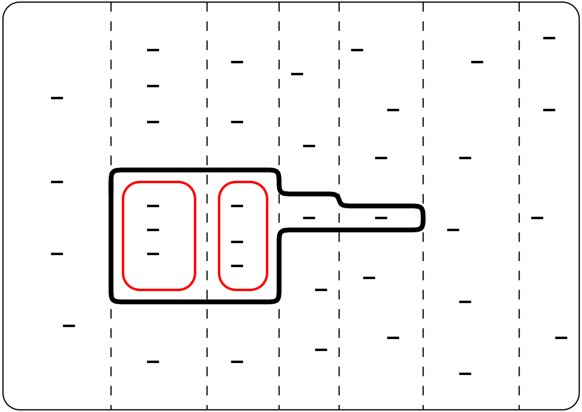
$e_3(x) = x$ for $x \in aN$ while $e_3(x) = a$ for $x \in N \cup a^2N$. Hence $e_3(e_3(x)) = e_3(x)$ and $U_3 = aN$ is a neighborhood.

$U_2 \notin \mathcal{N}_{\mathbf{A}}(\alpha)$ since U_2 does not meet any α-class nontrivially. $U_1 \in \mathcal{N}_{\mathbf{A}}(\alpha)$ but U_1 is not an α-minimal set because $U_3 \in \mathcal{N}_{\mathbf{A}}(\alpha)$ and $U_3 \subset U_1$. A computer can show that U_3 **is** an α-minimal set. In fact, the cosets of N are precisely the α-minimal sets.

Since each α-minimal set in this example is contained entirely inside an α-class, the α-traces and α-bodies are identical to the α-minimal sets, i.e., the cosets of N.

Warning: this is not the typical picture!!

[Typical picture on next page]



This portrays the classes of a minimal congruence α (dashed lines), one α-minimal set (dark black line), and an α-body consisting of two α-traces (red lines). The two points not in the body comprise the **tail**.

7. The 5-fold classification and types

Key step: focus on polynomial algebras induced on α-minimal sets and α-traces.

The latter are catalogued up to polynomial equivalence.

Theorem 1 (P. P. Pálfy)

Let $\mathbf{A}$ be a finite algebra, α a minimal congruence, and N an α-trace. Then the induced polynomial algebra $\mathbf{A}|_N$ is polynomially equivalent to one of:

1. a G-set.
2. a 1-dimensional vector space over a finite field.
3. a 2-element boolean algebra.
4. a 2-element lattice $(L, \{\wedge, \vee\})$.
5. a 2-element semilattice $(S, \vee)$.

Theorem 2

Let **A** be a finite algebra and α a minimal congruence. If N_1, N_2 are any two α-traces, then $\mathbf{A}|_{N_1} \cong \mathbf{A}|_{N_2}$.

Thus we get a 5-fold classification of minimal congruences.

Definition

For α a minimal congruence of **A**, the **type** of α is the common type

- Type 1 (**unary**)
- Type 2 (**vector space**)
- Type 3 (**boolean**)
- Type 4 (**lattice**)
- Type 5 (**semilattice**)

of the polynomial algebras induced on the α-traces of **A**.

Example: The minimal congruence of the group A_4 has "Type 2."

The 5 types reflect 5 distinct "local" structures in an algebra **A**.

In turn, the local structure reflects and is reflected by the global structure of **A**.

The lowest-order tool is:

Theorem 3

Let α be a minimal congruence of **A**.

(Connectedness) Each nontrivial α-block is the union of connected α-traces.

Moreover, **A** has enough unary polynomials to:

(Isomorphism) …map any α-trace isomorphically to any other.

(Density) …map any two distinct elements in an α-block to distinct elements of an α-trace.

Connecting local and global structure

Example:

Theorem

Let α be a minimal congruence of finite $\mathbf{A}$.

- α is abelian $\Leftrightarrow$ the type of α is 1 or 2.
- If α is abelian and $\mathbf{A}$ is idempotent, then each block of α (as a subalgebra of $\mathbf{A}$) is **quasi-affine** (i.e., is a subalgebra of a reduct of a module).

Typing $\mathrm{Con}(\mathbf{A})$

Suppose $\alpha \in \mathrm{Con}(\mathbf{A})$ is **not** minimal. Choose $\delta < \alpha$ so that α is minimal over δ.

[Picture]

Passing to $\mathbf{A}/\delta$, we can assign a type (1–5) to the pair (δ, α).

In this way, a type (1–5) is assigned to each edge of the graph of $\mathrm{Con}(\mathbf{A})$. Much is known.

In applications, one often needs local information about (δ, α) in **A** (not just in $\mathbf{A}/\delta$).

Leads to a refined def. of (δ, α)-minimal sets, traces and bodies ($\subseteq A$).

Polynomial algebras induced on (δ, α)-traces are completely understood in types 3 and 4, largely understood in cases 1, 2 and 5.

Classifying algebras by the types their varieties omit

Definition

Let $\mathbf{A}$ be a finite algebra and $i \in \{1, 2, 3, 4, 5\}$. We say that $\mathrm{var}(\mathbf{A})$

- **admits** type i if type i occurs in $\mathrm{Con}(\mathbf{B})$ for some (finite) $\mathbf{B} \in \mathrm{var}(\mathbf{A})$. (WLOG, $\mathbf{B} \leq \mathbf{A}^n$.)
- **omits** type i otherwise.

Omitting types gives us another way to classify $\mathrm{var}(\mathbf{A})$. For example:

Theorem

For $\mathbf{A}$ finite, TFAE:

- $\mathbf{A}$ has a Taylor term (equivalently, a weak NU-term).
- $\mathrm{var}(\mathbf{A})$ omits type 1.

Similar characterizations (via terms satisfying equations) exist for $\mathrm{var}(\mathbf{A})$ omitting any "down-set" of types (e.g., $\{1, 2\}$, $\{1, 5\}$, etc).

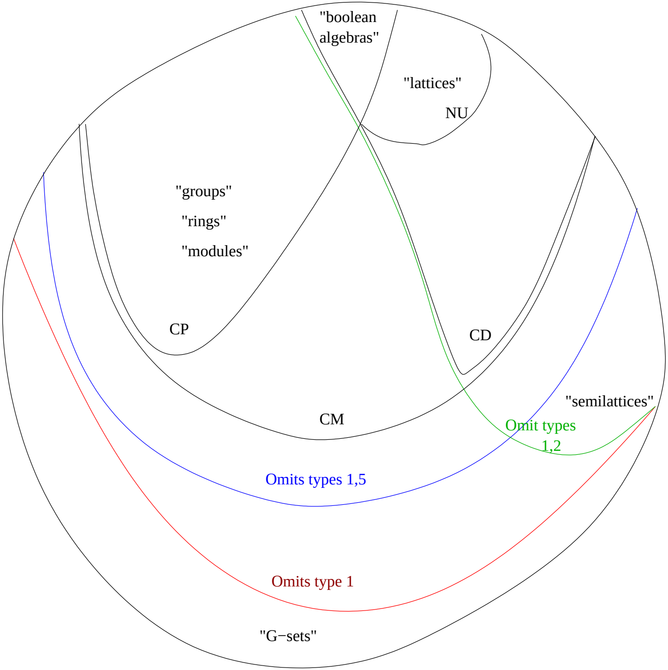
Fitting in the congruence classifications

Theorem

- $\mathrm{var}(\mathbf{A})$ is CM iff $\mathrm{var}(\mathbf{A})$ omits type 1 and 5 and has no tails.
- $\mathrm{var}(\mathbf{A})$ is CD iff $\mathrm{var}(\mathbf{A})$ omits type 1, 2 and 5 and has no tails.

Another relevant class is the class of $\mathbf{A}$ for which $\mathrm{var}(\mathbf{A})$ omits the abelian types 1,2. This class is characterized as those $\mathbf{A}$ for which every $\mathrm{Con}(\mathbf{B})$ satisfies a certain implicational law called SD($\wedge$).

[Big picture on next page]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Postscript (not included in lecture):

Tame congruence theory reveals how far is the gap between those idempotent **A** for which CSP(**A**) is known to be NP-complete (var(**A**) admits type 1), and those for which CSP(**A**) is known to be in P (CP, NU, bounded width, varieties generated by a finite conservative algebra).

Tame congruence theory suggests intermediate classes of algebras to be explore.

Under weak assumptions (e.g., that var(**A**) omits type 1), the theory yields subtle, positive structural information about all **B** $\in \mathrm{var}$(**A**).

Most importantly, the theory suggests one way to localize, divide and conquer the confusion of "all finite algebras."