Constraint Satisfaction Problems
A Survey

Ross Willard

University of Waterloo, CAN

Algebra & Algorithms
University of Colorado
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(with corrections)
Fix a finite algebra $A$.

**Definition**

A constraint network over $A$ is a pair $(n, \varphi)$ where

- $n \geq 1$
- $\varphi$ is a quantifier-free formula of the form $\bigwedge_{i \in I} R_i(x_i)$, where for each $i \in I$,
  - $x_i$ is a $d$-tuple of variables from $\{x_1, \ldots, x_n\}$ (for some $d$)
  - $R_i$ is a subuniverse of $A^d$.

The relation defined by $(n, \varphi)$ is

$$\text{Rel}_A(n, \varphi) = \{a \in A^n : \varphi(a)\}.$$
Example

Let \( A = (\{0, 1\}; x + y + z) \)
\( R_0 = \{(0, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0)\} \)
\( R_1 = \{(0, 0, 1), (0, 1, 0), (1, 0, 0), (1, 1, 1)\} \).

\( R_0, R_1 \leq A^3 \). Thus the following is a constraint network over \( A \):

\[
(6, R_0(x_1, x_2, x_3) \land R_1(x_1, x_4, x_5) \land R_0(x_2, x_4, x_6) \land R_1(x_3, x_5, x_6)) \).
\]

We can view \( \varphi \) as asserting (over \( \mathbb{Z}_2 \))

\[
\begin{align*}
x_1 + x_2 + x_3 &= 0 \\
x_1 + x_4 + x_5 &= 1 \\
x_2 + x_4 + x_6 &= 0 \\
+ x_3 + x_5 + x_6 &= 1.
\end{align*}
\]

\( \text{Rel}_A(6, \varphi) \) is the solution-set to this linear system.
Variant notations

A constraint network over $A$ is a pair $(n, \varphi)$, $\varphi = \bigwedge_i R_i(x_i)$ ...

<table>
<thead>
<tr>
<th>...</th>
<th>may be written as ...</th>
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<tbody>
<tr>
<td>$n$</td>
<td>${x_1, \ldots, x_n}$ ( = $V$, the set of variables)</td>
</tr>
<tr>
<td>$\varphi$</td>
<td>${(x_i, R_i) : i \in I}$ ( = $C$)</td>
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<tr>
<td></td>
<td>• $(x_i, R_i)$ is called a constraint</td>
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<td></td>
<td>• $x_i$ is its scope</td>
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<tr>
<td></td>
<td>• $R_i$ is its constraint relation</td>
</tr>
<tr>
<td>$(n, \varphi)$</td>
<td>$(V, C)$ or $(V, A, C)$</td>
</tr>
<tr>
<td>$\text{Rel}_A(n, \varphi)$</td>
<td>$\text{Sol}(V, C)$</td>
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</table>
Decision Problems

Definition

$(n, \varphi)$ is $k$-ary if each scope has length $\leq k$.

Definition

$\text{CSP}(A, k)$

- **Input:** A $k$-ary constraint network $(n, \varphi)$ over $A$.
- **Question:** Is $\text{Rel}_A(n, \varphi) \neq \emptyset$?

Dichotomy Conjecture (Feder & Vardi)

For all $A$ and $k$, $\text{CSP}(A, k)$ is in P or is NP-hard.

Algebraic Dichotomy Conjecture (Bulatov, Krokhin & Jeavons)

If $A$ has a Taylor operation, then $\text{CSP}(A, k)$ is in P for every $k$. $A$ is tractable.
Taylor operations

Definition
An operation $t : A^n \to A$ is a Taylor operation if

1. $t$ is idempotent ($t(x, x, \ldots, x) \approx x$);
2. For each $i = 1, \ldots, n$, $t$ satisfies an identity of the form $t(x) \approx t(y)$ with $x_i \neq y_i$.

Theorem (Taylor; Barto & Kozik; Hobby & McKenzie)
For a finite algebra $A$, the following are equivalent:

1. $A$ has a Taylor (term) operation.
2. $A$ satisfies some idempotent Maltsev condition not satisfied by $\text{SETS}$.
3. $A$ has an idempotent cyclic term $t(x_1, \ldots, x_n)$, i.e.,
   \[ t(x_1, x_2, \ldots, x_n) \approx t(x_2, \ldots, x_n, x_1). \]
4. $\forall(A)$ omits type 1.
Progress

Algebraic Dichotomy Conjecture

If \( A \) has a Taylor operation, then \( \text{CSP}(A, k) \) is in \( P \) for every \( k \).

\[ A \text{ is tractable} \]

Theorem

\( A \) is known to be tractable if:

1. \( V(A) \) is CM. (Dalmau ‘05 + IMMVW ‘07, using Barto ‘16?)
2. \( V(A) \) is SD(\( \land \)). (Barto & Kozik ‘09; Bulatov ‘09)
3. \( A \) is Taylor + conservative, i.e. \( Su(A) = P(A) \). (Bulatov ‘03)
4. \( A \) is Taylor and \( |A| = 2 \) or 3. (Schaefer ‘78, Bulatov ‘02)
Definition
Let $A$ be a finite algebra, $\mathcal{A}$ a set of finite algebras.

1. $\text{CSP}(A) = \bigcup_k \text{CSP}(A, k)$. "Global"
2. $\text{CSP}(\mathcal{A}, k) = \bigcup_{A \in \mathcal{A}} \text{CSP}(A, k)$. "Uniform"

Can’t ask these problems to be in P. (Set of inputs is problematic.)

Definition
Say $\text{CSP}(A)$ [$\text{CSP}(\mathcal{A}, k)$] is "in" P if there is a poly-time algorithm which correctly decides all inputs to $\text{CSP}(A)$ [$\text{CSP}(\mathcal{A}, k)$].

Global Tractability Problem
If $A$ is tractable, does it follow that $\underbrace{\text{CSP}(A)}_{\text{A is globally tractable}}$ is "in" P?

Uniform Tractability Question
(For a given Taylor class $\mathcal{A}$): Is $\underbrace{\text{CSP}(\mathcal{A}, k)}_{\text{A is uniformly tractable}}$ "in" P for all $k$?
**Theorem**

A is known to be globally tractable if:

1. A has a **cube term**. (Dalmau ‘05 + IMMVW ‘07)
2. \( V(A) \) is \( SD(\wedge) \). (Bulatov ‘09; Barto ‘14)
3. A is Taylor + conservative. (Bulatov ‘03)
4. A is Taylor and \( |A| = 2 \) or 3. (Schaefer ‘78, Bulatov ‘02)

**Theorem (Bulatov ‘09; Barto ‘14)**

The class \( SD_\wedge \) of all finite algebras generating an \( SD(\wedge) \) variety is uniformly globally tractable.
Open problems

1. If $V(A)$ is congruence modular, is $A$ globally tractable?

2. Is the class $\mathcal{M}$ of finite Maltsev algebras uniformly tractable?

3. If $A$ has a difference term, is $A$ tractable?

4. Suppose $A$ is idempotent and has a congruence $\theta$ such that
   - $A/\theta \in SD_\wedge$, and
   - Each $\theta$-block is in $\mathcal{M}$.
   ("SD($\wedge$) over Maltsev.") Is $A$ tractable?
Standard reductions

\[ \text{CSP}(A, k) \text{ reduces to:} \]

1. CSP(\(A\|_U, k\)), where \(U\) is a minimal range of a unary idempotent term, and \(A\|_U\) is the induced term-minimal algebra defined on \(U\).

2. CSP((\(A\|_U\)\text{id}, k)) where \((B)^{\text{id}}\) is the idempotent reduct of \(B\).
   (This is the “reduction to the idempotent case.”)

3. CSP(\(A^{\lceil k/2 \rceil}, 2\))

4. multi-CSP(\(H(A)_{si}, kd\)), where \(A\) is a subdirect product of \(d\) subdirectly irreducible homomorphic images.

5. CSP(\(A^+, k\)) where \(A^+ = (A; \text{Pol}(\text{Su}(A^k)))\).
Conditioning the input – local consistency

Let \((n, \varphi)\) be a 2-ary constraint network over \(A\).

At essentially no cost, one can assume that \((n, \varphi)\) is “determined” by a “(2,3)-minimal” constraint network.

Definition
A 2-ary constraint network \((n, \varphi)\) is a \((2,3)\)-system\(^1\) provided for all \(i, j \in \{1, 2, \ldots, n\}\):

1. \(\varphi\) has exactly one constraint \(R_{i,j}(x_i, x_j)\) with scope \((x_i, x_j)\).
2. \(R_{j,i} = (R_{i,j})^{-1}\).
3. For all \(k\), \(R_{i,j} \subseteq R_{i,k} \circ R_{k,j}\).

The “associated potatoes” are \(A_i := \text{proj}_1(R_{i,j}), \ i = 1, \ldots, n\).

Fact
There is a poly-time algorithm which, given a 2-ary constraint network over \(A\), outputs an equivalent \((2,3)\)-system over \(A\).

\(^1\)There is no standard terminology.
Conditioning the input – absorption

Definition
Suppose $A$ is a finite idempotent algebra and $B \leq A$.

1. $B$ is an **absorbing subalgebra** if there exists a term operation $t(x_1, \ldots, x_m)$ of $A$ such that

   $$t(B, \ldots, B, A, B, \ldots, B) \subseteq B$$

   for all possible positions of $A$.

2. $A$ is **absorption-free** if it has no proper absorbing subalgebra.

Given a $(2,3)$-system $(n, \varphi)$ over an idempotent $A$, Barto & Kozik show how to “shrink” the associated potatoes to absorption-free algebras, though losing $(2,3)$-systemhood and equivalency.

In some situations this has proven to be useful.
Miklós magic

Lemma (Maróti ‘09)

Suppose $A$ is idempotent and has a term operation $t(x, y)$ such that:

1. $A \models t(x, t(x, y)) \approx t(x, y)$.
2. $t(a, x)$ is non-surjective, for all $a \in A$.
3. There exists a proper subalgebra $C < A$ such that if $t(x, a)$ is surjective then $a \in C$.

Then CSP($A, k$) can be reduced to multi-CSP($B \setminus \{A\}, \ell$), where

- $B$ is the closure of $\{A\}$ under $H$, $S$, and “idempotent unary polynomial retracts.”
- $\ell = \max(k, |A|)$.

This may seem random, but it is useful (and the proof is beautiful).
Moving forward

Suppose \((n, \varphi)\) is a \(k\)-ary constraint network over \(A\), and \(R = \text{Rel}_A(n, \varphi) \leq A^n\).

Definition

A **compact** \(k\)-frame for \(R\) is a subset \(F \subseteq R\) such that

1. \(\text{proj}_J(F) = \text{proj}_J(R)\) for all \(J \subseteq \{1 \ldots, n\}\) with \(|J| \leq k\).
2. \(|F| \leq |A|^k \cdot \binom{n}{k}\).

Every relation definable by a \(k\)-ary constraint network over \(A\) has a compact \(k\)-frame, and is determined by any one of its \(k\)-frames.

**Speculation:** Is it possible to mimic the few subpowers algorithm without having few subpowers?
To carry this out, we would need a notion of “compact \( k \)-representation” extending compact \( k \)-frames with more data.

The following problem seems central:

**Functional Dependency Problem**

Suppose

- \( A \) is finite, idempotent, Taylor.
- \( F \) is a compact \( k \)-frame for a relation \( R \preceq A^n \) defined by some \( k \)-ary constraint network over \( A \).
- \( X \subseteq \{1, \ldots, n\} \) and \( \ell \in \{1, \ldots, n\} \setminus X \).

What additional data would enable us to efficiently decide whether \( \text{proj}_{X \cup \{\ell\}}(R) \) is the graph of a function \( f : \text{proj}_X(R) \to \text{proj}_\ell(R) \)?
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