The decidable discriminator variety problem

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Variations on Homogeneity
In the box: certain $\forall_1$ classes of structures which are

- locally finite
- in a finite signature

”small $\forall_1$ classes”

Which small $\forall_1$ classes are in the box?
Hints

Which small $\forall_1 \mathcal{K}$ are in the box?

1. If $\mathcal{K}$ is a finite set of finite structures, then $\mathcal{K}$ is in the box.
2. If every countable member of $\mathcal{K}$ is (hereditarily) homogeneous, then $\mathcal{K}$ is in the box.

▶ homogeneous: every isomorphism between finite substructures extends to an automorphism.
▶ hereditarily: every substructure is homogeneous.

3. The box is a candidate for the smallest “natural” collection of small $\forall_1$ classes satisfying (1)–(2).

Intuition

The box captures some version of “hereditarily homogeneous modulo finite.”
Guess #1

Definition

1. \( M \) is **weakly hereditarily homogeneous** if there exists a finite set \( A \subseteq M \) such that \( M_A \) is hereditarily homogeneous.

2. A small \( \forall_1 \) class \( \mathcal{K} \) is **weakly hereditarily homogeneous** if there exists \( n \geq 0 \) such that every countable member \( M \in \mathcal{K} \) is weakly hereditarily homogeneous via a set \( A \subseteq M \) of size \( \leq n \).

Getting warm!

- Every class in the box is weakly hereditarily homogeneous.
- But not conversely: the class 

  \[ \{ \text{graphs having at most one edge} \} \]

  is **not** in the box.
Guess #2

Definition

A small $\forall_1$ class $\mathcal{K}$ is upwardly weakly hereditarily homogeneous if there exists $n \geq 0$ such that for all $M \in \mathcal{K}_{\text{fin}}$ there exists $A \subseteq M$ with $|A| \leq n$, satisfying:

1. $M_A$ is hereditarily homogeneous.
2. For all $N \in \mathcal{K}_{\text{fin}}$ and embeddings $\sigma_1, \sigma_2 : M \hookrightarrow N$ with $\sigma_1|_A = \sigma_2|_A$, there exists $\alpha \in \text{Aut } N$ with $\alpha \circ \sigma_1 = \sigma_2$.

Getting hot!!

- $\{\text{graphs with } \leq 1 \text{ edge}\}$ is not UWHH.
- Every class in the box is UWHH.
- (I don’t know if the converse holds.)
Suppose $\kappa$ is a small $\forall_1$ class.

**Definition**

$\kappa$ is **in the box** if there exists a relation $\triangleleft$ between finite sets and members of $\kappa_{fin}$ such that for some $n \geq 0$,

1. $A \triangleleft M$ implies $A \subseteq M$, $M_A$ is homogeneous, and $|A| \leq n$.
2. $\triangleleft$ is invariant under isomorphisms.
3. For all $M \in \kappa_{fin}$ there exists $A \triangleleft M$.
4. If $A \triangleleft M$ and $A \subseteq M' \leq M$, then $A \triangleleft M'$.
5. If $A \triangleleft M \leq N \in \kappa_{fin}$ then there exists $B \triangleleft N$ with $A \subseteq B$.
6. If $A \subseteq B \triangleleft N$ and $M_1, M_2 \leq N$ with $A \triangleleft M_1, M_2$, then every isomorphism $\sigma : M_1 \cong M_2$ fixing $A$ pointwise extends to some $\alpha \in \text{Aut } N$ fixing $B$ pointwise.

(Ugh)
Decidable equational classes
Universal algebra

**Algebraic structure**, or **algebra**: a structure in a signature with no relation symbols.

**Equational theory**: a deduction-closed set of **identities**

\[ \forall x : s(x) = t(x) \]

**Equational class**: \( \text{Mod}(T) \) for some equational theory \( T \).
Decidable Equational Class Problem

**Problem**
For which equational classes $\mathcal{E}$ in finite signature is the 1st-order theory of $\mathcal{E}$ decidable?

**Theorem (McKenzie, Valeriote 1989)**
In the locally finite case, this problem is solved modulo two special cases:

1. Modules over a finite ring.
2. “Discriminator varieties.”

What is a *discriminator variety*?
Discriminator varieties
Recipe

1. Start with a $\forall_1$-class of structures.

2. Replace each $n$-ary basic relation $R$ with an $n + 2$-ary operation $f_R$ defined by
   
   \[ f_R(x, y, z) = \begin{cases} 
   y & \text{if } R(x) \\
   z & \text{else.} 
   \end{cases} \]

3. Also add $f_\equiv$.

4. Denote the resulting $\forall_1$-class of algebras $\mathcal{K}^*$.

5. Let $T_e(\mathcal{K}^*)$ be the equational theory of $\mathcal{K}^*$.

6. $\mathcal{D}(\mathcal{K}) := \text{Mod}(T_e(\mathcal{K}^*))$ is a typical discriminator variety.

   - Note: $\mathcal{K}^*$ is the class of simple algebras in $\mathcal{D}(\mathcal{K})$. 

Example

Start with $\mathcal{K} = \{2\}$ where $2 = (\{0, 1\}, 0, 1)$.

$\mathcal{K}^* = \{2^*\}$ where $2^* = (\{0, 1\}, f_\equiv, 0, 1)$,

$$f_\equiv(x, y, z, w) = \begin{cases} z & \text{if } x = y \\ w & \text{else.} \end{cases}$$

Note: $2^*$ is the 2-element boolean algebra. Hence

\[
\mathcal{D}(\{2\}) = \text{Mod}(T_e(\{2^*\})) \\
= \text{Mod}(T_e(\{\text{the 2-element boolean algebra}\})) \\
= \{\text{all boolean algebras}\}
\]
1. Discriminator varieties correspond to $\forall_1$ classes:

\[
(\text{loc. fin., fin. sign.}) \leftrightarrow \forall_1 \text{ classes}
\]

\[
\mathcal{D}(\mathcal{K}) \iff \mathcal{K}^* \equiv \mathcal{K}
\]

2. Discriminator varieties are (equational) classes of “generalized boolean algebras.”
The Decidable Discriminator Variety problem

The question

Which (loc. fin., fin. sign.) discriminator varieties have decidable 1st-order theory?

can be reformulated

Which (small) $\forall_1$ classes $\mathcal{K}$ are such that $D(\mathcal{K})$ has decidable 1st-order theory?

Conjecture

Answer to 2nd question: the ones in the box!
Evidence

Theorem (W)

Suppose $\mathcal{K}$ is in the box.

1. $\{\text{graphs}\}$ does not interpret\(^1\) into $\mathcal{D}(\mathcal{K})$.

2. If $\text{Th}_{\forall_1}(\mathcal{K})$ is decidable (e.g., if $\mathcal{K}$ is finitely axiomatizable), then $\text{Th}(\mathcal{D}(\mathcal{K}))$ is decidable.

Moreover

In classes studied to date\(^2\), no counter-examples found to:

1. $\mathcal{K}$ not in the box $\Rightarrow$ $\{\text{graphs}\}$ interprets into $\mathcal{D}(\mathcal{K})$.

2. $\mathcal{K}$ in the box $\Rightarrow$ $\mathcal{K}$ finitely axiomatizable.

\(^1\)“right totally” as per Hodges

\(^2\)unary algebras (W ’93), lattices (W ’94), dihedral groups (Delić ‘05)
Ingredients in the proof

- Every member of $\mathcal{D}(\mathcal{K})$ has a representation as the algebra of global sections of some Hausdorff sheaf over a Stone space, with stalks from $\mathcal{K}^*$.

- Assuming $\mathcal{K}$ is in the box, one can obtain a (non-effective) Feferman-Vaught analysis of the countable members of $\mathcal{D}(\mathcal{K})$ (via their representations).

- This translates the theory of $\mathcal{D}(\mathcal{K})$ to the theory of boolean algebras with countably many ideals (decidable by Rabin).

- If $\text{Th}_{\forall_1}(\mathcal{K})$ is decidable, then the translation can be made effective.
Help!

Recall: \( \mathcal{K} \) in the box \( \implies \mathcal{K} \) UWHH.

1. Does \( \mathcal{K} \) UWHH \( \implies \) \( \mathcal{K} \) in the box?

2. What are generic obstacles to UWHH? To being in the box?
   ▶ In all examples I know, there is a witnessing pair \( M < N \) of countably infinite structures and a finite set \( A \) such that \( \text{Aut}(M_A) \) has an infinite orbit that gets “badly split” in \( N_A \).

3. Does UWHH (or being in the box) imply finite axiomatizability?

4. Does anyone give a rip??

Thank you!